Exchangeable Equilibria

N. D. Stein   A. Ozdaglar   P. A. Parrilo

Laboratory for Information and Decision Systems
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

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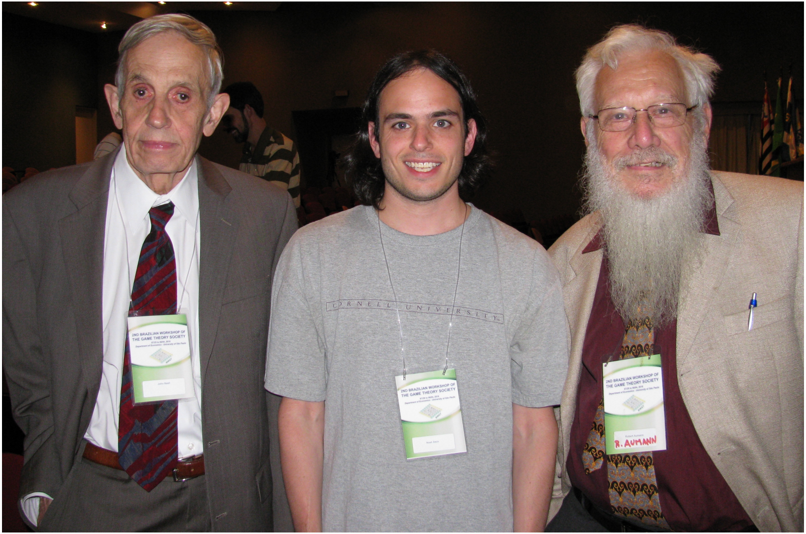
~~Dramatis~~ Ludorum Personae



$$\mathrm{NE}_{\mathrm{Sym}} \subseteq \mathrm{XE}_{\mathrm{Sym}} \subseteq \mathrm{CE}_{\mathrm{Sym}}$$

Introduction

Apology

- Major error in last year's talk remains
- Thanks to Sergiu Hart and Eran Shmaya for finding it

Outline

- Correlated equilibria in finite games
- Correlation schemes
- Symmetric games
- Exchangeable equilibria
- Examples
- Properties

Setting

Finite games

- Finite number of players
- Each player has a finite set of actions
- Each player has a utility function mapping action profiles to reals

Chicken – Nash equilibria

The game of chicken

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4, 4)$	$(1, 5)$
Macho	$(5, 1)$	$(0, 0)$

Nash equilibria (NE)

- All three equilibria in three notations

Tuple	(M, W)	(W, M)	$(\frac{1}{2}W + \frac{1}{2}M, \frac{1}{2}W + \frac{1}{2}M)$
Product	$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
Joint law	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Chicken – correlated equilibria

The game of chicken

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4, 4)$	$(1, 5)$
Macho	$(5, 1)$	$(0, 0)$

(Internal) correlated equilibria (CE)

- Example correlated equilibria (joint laws)

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Joint distributions of actions

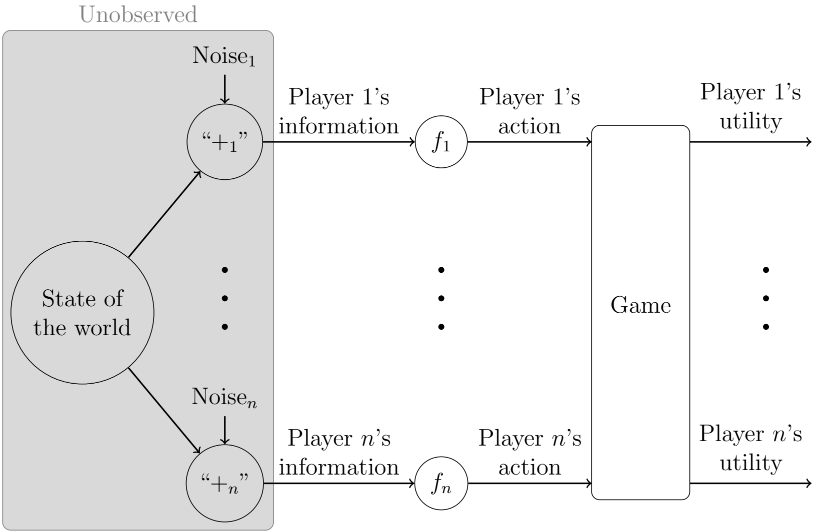
How does correlation arise?

- Assumptions on information available to players have implications for how their actions are jointly distributed
- Private randomization $\rightarrow$ independent
- Trusted mediator $\rightarrow$ arbitrary distributions
- Noisy observations of environment (sunspots, etc.) $\rightarrow$?

Plan

- Investigate this final assumption (in a symmetric setting)
- Need a way to talk about such information assumptions
- Various essentially equivalent choices of formalism

Correlation schemes



Correlation schemes

External viewpoint

- Model generation of distribution of actions
- Players' information is independent given hidden state
- Players choose f_i: action as a function of information
- Def.: **external correlated equilibrium** $\Leftrightarrow f_1, \dots, f_n$ are a pure Nash equilibrium of induced game
  - Corollary: induced distribution of actions is an internal correlated equilibrium

Examples in this framework

- Nash equilibria: fixed state, private randomization only
- Internal correlated equilibria: state is random action profile, noise is forgetting others' strategies

Symmetric setting

Symmetric games

- Finite number of players
- Each player has the same finite set of actions
- Permuting players permutes utilities in the same way
  - Doesn't matter how players are labeled
- e.g. symmetric bimatrix games: $B = A^T$
- From now on, all games are symmetric

Symmetric correlated equilibria without a mediator

Clone assumption

- View players as independent instances of identical decision-making agents
- Each faces the same situation
- Each makes noisy measurements of the same aspects of the environment
- Under same information, each chooses same action

Definition

- A **symmetric external correlated equilibrium** is an external correlated equilibrium such that information is i.i.d. conditioned on state and $f_1 = \cdots = f_n$.

Symmetric external correlated equilibria

The main question

- Which internal correlated equilibria correspond to action distributions of symmetric external correlated equilibria?

First steps

- Example: symmetric Nash equilibria
  - In Chicken: $\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$
- Not all correlated equilibria are of this form: must be symmetric (invariant to relabeling players)
  - Rules out e.g. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in Chicken
- What about $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$?

Exchangeable equilibria

Definition

- An **exchangeable equilibrium** (XE_{Sym}) is the action distribution of a symmetric external correlated equilibrium.

The main question, rephrased

- Which symmetric (internal) correlated equilibria are exchangeable?

Properties

- Exchangeable equilibria are the internal correlated equilibria which are i.i.d. conditioned on a hidden state
- Without loss of generality state takes finitely many values
- $\mathrm{conv}(NE_{Sym}) \subseteq XE_{Sym} \subseteq CE_{Sym}$

Aside on naming

Exchangeability

- Def.: random variables $X_1, X_2, \ldots$ are **exchangeable** $\equiv$ distribution invariant under permuting variables
- $X_1, \ldots, X_n$ conditionally i.i.d. $\Leftrightarrow$ extendable to an exchangeable sequence $X_1, X_2, \ldots$ (De Finetti's Theorem)
- Leads to other interpretations of exchangeable equilibria
- Will not discuss these today

Special properties

- Exchangeable equilibria are the internal correlated equilibria of the form $W = \sum_{j=1}^{k} \lambda_j x_j x_j^T$
  - λ, x_j are probability column vectors
- e.g., weather is good ('☼') or bad ('☹') with equal probability. Players' independent atmospheric measurements yield true state or '?' with equal probability.

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{matrix}☼\\ ☹\end{matrix}\begin{bmatrix}1/2\\ 1/2\end{bmatrix}, \qquad x_1 = \begin{matrix}☼\\ ?\\ ☹\end{matrix}\begin{bmatrix}1/2\\ 1/2\\ 0\end{bmatrix}, \qquad x_2 = \begin{matrix}☼\\ ?\\ ☹\end{matrix}\begin{bmatrix}0\\ 1/2\\ 1/2\end{bmatrix}$$

$$\begin{matrix} & ☼ & ? & ☹ \\ ☼ & 1/8 & 1/8 & 0 \\ ? & 1/8 & 1/4 & 1/8 \\ ☹ & 0 & 1/8 & 1/8 \end{matrix} = \frac{1}{2}\begin{bmatrix}1/2\\ 1/2\\ 0\end{bmatrix}\begin{bmatrix}1/2\\ 1/2\\ 0\end{bmatrix}^T + \frac{1}{2}\begin{bmatrix}0\\ 1/2\\ 1/2\end{bmatrix}\begin{bmatrix}0\\ 1/2\\ 1/2\end{bmatrix}^T$$

The bimatrix case (continued)

Necessary conditions for $W = \sum_{j=1}^{k} \lambda_j x_j x_j^T$

- $\text{Tr}(W) = \sum_{j=1}^{k} \lambda_j \|x_j\|_2^2 > 0$
  - Rules out $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ in Chicken
  - Can have $\text{XE}_{\text{Sym}} \subsetneq \text{CE}_{\text{Sym}}$
- Symmetric, nonnegative elements, positive semidefinite
  - These necessary conditions are sufficient in bimatrix games with at most 4 strategies per player [corollary of Maxfield and Minc]

Example #1

Chicken

- Utilities $A = B^T = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$
- Exchangeable equilibria are symmetric matrices $\begin{bmatrix} p & q \\ q & r \end{bmatrix}$
- $p, q, r \geq 0$, $p + 2q + r = 1$
- Incentive constraints: $q \geq p, r$
- Semidefiniteness: $pr \geq q^2$
- Algebra $\ldots p = q = r = 1/4$
- $\mathrm{XE}_{\mathrm{Sym}} = \mathrm{NE}_{\mathrm{Sym}} = \left\{ \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \right\}$
- $\mathrm{conv}(\mathrm{NE}_{\mathrm{Sym}}) = \mathrm{XE}_{\mathrm{Sym}} \subsetneq \mathrm{CE}_{\mathrm{Sym}}$

Example #2

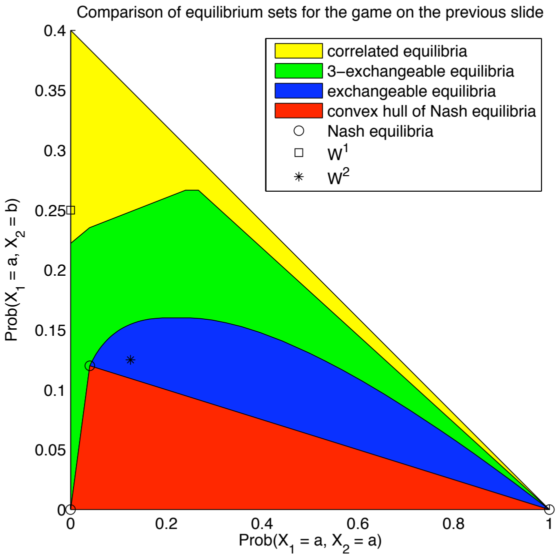
Game with $\text{conv}(\text{NE}_{\text{Sym}}) \subsetneq \text{XE}_{\text{Sym}} \subsetneq \text{CE}_{\text{Sym}}$

(u_1, u_2)	a	b	c
a	$(5,5)$	$(5,4)$	$(0,0)$
b	$(4,5)$	$(4,4)$	$(4,5)$
c	$(0,0)$	$(5,4)$	$(5,5)$

- Symmetric Nash equilibria:
  $\begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1/5 & 3/5 & 1/5\end{bmatrix}$
- Non-exchangeable correlated equil.: $W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$
- Exchangeable equilibrium not in $\text{conv}(\text{NE}_{\text{Sym}})$:

$$W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}^T + \frac{1}{2} \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^T$$

Example #2, plotted



- Correlated equil. which is not exchangeable:

$$W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

- Exchangeable equil. not in conv(Nash):

$$W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

Example #3

Don't be greedy

- $C_1 = C_2 = C_3 = \{0, 1\}$
- Common utility function

$$u(s_1, s_2, s_3) = \begin{cases} 0, & \text{when } s_1 = s_2 = s_3 = 1 \\ s_1 + s_2 + s_3, & \text{otherwise} \end{cases}$$

- Symmetric Nash equilibria are i.i.d. Bernoulli(p) for some p
- Algebra: only solution is $p = \frac{1}{\sqrt{3}}$
- Semidefiniteness characterization of exchangeability as in bimatrix case
- More algebra: $\mathrm{XE}_{\mathrm{Sym}} = \mathrm{NE}_{\mathrm{Sym}}$
- Unique exchangeable equilibrium, irrational probabilities

Example #4

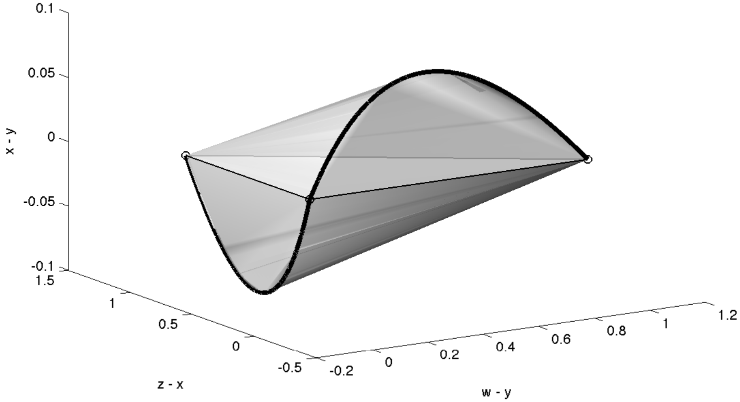
Three-player coordination game

- $C_1 = C_2 = C_3 = \{0, 1\}$
- Common utility function

$$u(s_1, s_2, s_3) = \begin{cases} 1, & \text{when } s_1 = s_2 = s_3 \\ 0, & \text{otherwise} \end{cases}$$

- $\text{NE}_{\text{Sym}} = \left\{\text{i.i.d. Bernoulli}(p) \mid p = 0, \frac{1}{2}, 1\right\}$
- Let $\pi_\lambda =$ i.i.d. Bernoulli$\left(\frac{1}{3}\right)$ with probability λ, else i.i.d. Bernoulli$\left(\frac{2}{3}\right)$
- $\text{conv}(\text{NE}_{\text{Sym}}) \ni \pi_\lambda$ if and only if $\lambda = \frac{1}{2}$
- $\text{XE}_{\text{Sym}} \setminus \text{conv}(\text{NE}_{\text{Sym}}) \ni \pi_\lambda$ for $\frac{1}{3} \leq \lambda < \frac{1}{2}$ and $\frac{1}{2} < \lambda \leq \frac{2}{3}$
- $\text{CE}_{\text{Sym}} \setminus \text{XE}_{\text{Sym}} \ni$ uniform distribution over $\{s \mid \sum_i s_i \neq 2\}$

Example #4, plotted



NE_{Sym} = circles
XE_{Sym} = convex hull of arcs joining circles (not polyhedral)
CE_{Sym} = polyhedron containing XE_{Sym} (omitted)

The set of exchangeable equilibria

Geometric properties of XE_{Sym}

- Compact, convex
- Semialgebraic, not necessarily polyhedral
- Contains convex hull of symmetric Nash equilibria
  - Equal in 2×2 case
  - Can be distinct with more strategies or players
- Contained in symmetric correlated equilibria
  - Strict containment if asymmetric Nash equilibrium exists
- Nonempty
  - Proof: symmetric version of Nash's theorem
  - Fixed-point-free proof: symmetrize Hart & Schmeidler 1989
- Bimatrix case only: contains a rational element
  - Proof: look at symmetric Nash equilibria
  - Can fail in multiplayer case

Wrap-up

Summary

- Main contribution: exchangeable equilibria (XE_{Sym})
- Various game-theoretic interpretations
- Sandwiched between symmetric Nash and corr. equil.

Not discussed

- Computable to within ϵ in time poly(input size, $\log \frac{1}{\epsilon}$)
- Generalizations "closer to" conv(NE_{Sym})
- Broader notions of symmetry
- Exchangeable equilibria in asymmetric games

Open questions

- Convexity-based proof of Nash's theorem
- Computational complexity of exact exchangeable equilibria