A Fixed Point Free Proof of Nash's Theorem via Exchangeable Equilibria

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Main ideas

"Bonus" symmetry

- e.g. n player game invariant under cyclic shifting of players
- Invariant mixed Nash equilibrium $(\pi_1, \ldots, \pi_n)$
- $\pi_1 = \pi_2, \pi_2 = \pi_3, \ldots, \pi_{n-1} = \pi_n, \pi_n = \pi_1$
- $(\pi_1, \ldots, \pi_n)$ invariant under arbitrary permutations

Elementary existence proofs

- Structure of game $\rightsquigarrow$ structure of Nash equilibria (NE)
- e.g. for games in some class, $\text{NE} \cap \Sigma \neq \emptyset$
- Set CE of correlated equilibria is convex, $\text{NE} \subset \text{CE}$
- $\text{NE} \cap \Sigma \subset \text{conv}(\text{NE} \cap \Sigma) \subset \text{CE} \cap \text{conv}(\Sigma)$
- Elementary proof that last set is nonempty

Outline

Background

- Games
- Nash and correlated equilibria
- Symmetries
- Hart and Schmeidler's proof of existence of CE

The proof

- Carefully choose classes of games and sets Σ
- Mimic HS proof to show nonemptiness of $CE \cap \text{conv}(\Sigma)$
- Repeat
- Limiting argument gives Nash existence

Games

Definition

A **finite game (in strategic form)** consists of n players, each with a finite **pure strategy set** C_i and a **utility function** $u_i : C \to \mathbb{R}$ where $C := C_1 \times \cdots \times C_n$.

Notation

- $\Gamma :=$ a game
- $\Delta(C_i) :=$ probability distributions on $C_i =$ mixed strategies
- $\Delta := \Delta(C) =$ correlated strategies
- $\Delta^\Pi := \Delta(C_1) \times \cdots \times \Delta(C_n) =$ strategy profiles $\subset \Delta$

Equilibria

Definition

An ϵ**-Nash equilibrium** is a mixed strategy profile $(\pi_1, \ldots, \pi_n) \in \Delta^\Pi$ such that $u_i(s_i, \pi_{-i}) \leq u_i(\pi) + \epsilon$ for all i and $s_i \in C_i$. The set of such is ϵNE. The case $\epsilon = 0$ defines the set NE of **Nash equilibria**.

Definition

A **correlated equilibrium** is a joint distribution $\pi \in \Delta$ such that if $(X_1, \ldots, X_n)$ are jointly distributed according to π then X_i is almost surely a best response to the random conditional distribution $\mathbb{P}(X_{-i} \mid X_i)$. The set of such is CE.

Symmetries

Definition

A **symmetry** σ of a game has two pieces:

- Permutation of the set of players and
- Permutation of the disjoint union of the strategy sets,

which are compatible with each other:

- Image $\sigma(C_i) = C_{\sigma(i)}$

and leave the utilities invariant.

Groups

Groups of symmetries

- Composition and inverse of symmetries are symmetries
- We usually speak of a (finite) group G of symmetries
- This is merely language; no group theory used

Notation

- $S_n :=$ **symmetric group on n letters** $=$ permutations of $\{0, \ldots, n-1\}$
- $\mathbb{Z}_n :=$ **cyclic group of order n** $=$ permutations of the form $m \mapsto m + r \bmod n$

Examples of symmetric games

Symmetric bimatrix games

- A and B payoff to row and column players
- Symmetry under player swap: A, B square, $B = A'$
- e.g., chicken, prisoner's dilemma, stag hunt, etc.
- Symmetry group $\mathbb{Z}_2$

An n-player anti-coordination game

- $C_1 = C_2 = \ldots = C_n$
- $u_i(s) = \begin{cases} 1, & s_i \neq s_{i+1} \\ 0, & \text{else} \end{cases}$ subscripts interpreted $\mod n$
- Invariant under cyclic group $\mathbb{Z}_n$ permuting players

Symmetric distributions

Properties

- Symmetry σ maps distribution $\pi \in \Delta$ to $\sigma_*(\pi) \in \Delta$
- σ_* preserves structure: Δ^Π, NE, CE
- We say π is **symmetric** if $\sigma_*(\pi) = \pi$ for all $\sigma \in G$
- Sets of symmetric distributions: $\text{CE}_G \subseteq \Delta_G$, $\text{NE}_G \subseteq \Delta^\Pi_G$

Example: *n*-player anti-coordination game

- $\pi \in \Delta^\Pi_{\mathbb{Z}_n} \Rightarrow \pi$ i.i.d. $\Rightarrow \pi$ invariant under all permutations
- $\Delta^\Pi_{\mathbb{Z}_n} = \Delta^\Pi_{S_n} \subsetneq \Delta_{S_n} \subsetneq \Delta_{\mathbb{Z}_n}$
- $\Delta^\Pi_{\mathbb{Z}_n}$ is "more symmetric" than $\Delta_{\mathbb{Z}_n}$

Our goal

Nash's Theorem

For any game with symmetry group G, $\mathrm{NE}_G \neq \emptyset$.

Goal

- Prove this
- Without using fixed point theorems
- We would settle for the nonsymmetric version $\mathrm{NE} \neq \emptyset$
- The proof gives the symmetric version automatically

Symmetric correlated equilibria

Theorem (Hart and Schmeidler, Nau and McCardle)

For any game, CE $\neq \emptyset$*.*

Proof.

- Wait a few slides. □

Corollary

For any game with symmetry group G, $\text{CE}_G \neq \emptyset$*.*

Proof.

- Let $\pi \in \text{CE} \implies \sigma_*(\pi) \in \text{CE}$ for all $\sigma \in G$
- Average $\frac{1}{|G|} \sum_{\sigma \in G} \sigma_*(\pi) \in \Delta_G \cap \text{CE}$ by convexity □

Exchangeable distributions

Natural question

- Nash's theorem gives us equilibria with "bonus" symmetry
- Proof there are correlated equilibria with such symmetry?
  - Averaging fails

Definition

The **exchangable distributions** are $\Delta_G^X := \text{conv}(\Delta_G^\Pi)$.

Example: *n*-player anti-coordination game

- $\Delta_{\mathbb{Z}_n}^\Pi \subset \Delta_{S_n}$
- $\Delta_{\mathbb{Z}_n}^X = \text{conv}(\Delta_{\mathbb{Z}_n}^\Pi) \subseteq \Delta_{S_n} \subsetneq \Delta_{\mathbb{Z}_n}$

Exchangeable distributions, continued

Example: symmetric bimatrix games

- $\Delta = \{m \times m \text{ probability matrices}\}$
- $\Delta_{\mathbb{Z}_2} = \{m \times m \text{ symmetric probability matrices}\}$
- $\Delta^{\Pi} = \{xy' \mid x, y \text{ probability column vectors}\}$
- $\Delta^{\Pi}_{\mathbb{Z}_2} = \{xx' \mid x \text{ a probability column vector}\}$
- $\Delta^{X}_{\mathbb{Z}_2} = \text{conv}(\Delta^{\Pi}_{\mathbb{Z}_2}) =$ **completely positive** prob. mat.
- Elements of $\Delta^{\Pi}_{\mathbb{Z}_2}$, hence $\Delta^{X}_{\mathbb{Z}_2}$ are positive semidefinite
- Those in $\Delta_{\mathbb{Z}_2}$ need not be
  - e.g. $\det \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} = -0.25$
- $\Delta^{X}_{\mathbb{Z}_2} \subsetneq \Delta_{\mathbb{Z}_2}$

Exchangeable equilibria

Definition

The set of (G-)**exchangeable equilibria** is $\mathrm{XE}_G := \mathrm{CE} \cap \Delta_G^X$.

Remarks

- Exchangeable equilibria are correlated equilibria having all the "bonus" symmetry of the symmetric Nash equilibria
- XE_G is convex and compact.
- $\mathrm{conv}(\mathrm{NE}_G) \subset \mathrm{XE}_G \subset \mathrm{CE}_G$
- Inclusions can be strict (even in symmetric bimatrix case)
- Proving $\mathrm{XE}_G \neq \emptyset$ does not prove $\mathrm{NE}_G \neq \emptyset$
- It is an important step, can be done by tweaking HS proof

Hart and Schmeidler's proof

Theorem (HS 1989, NM 1990 is similar)

For any game, CE $\neq \emptyset$.

Proof.

- Given Γ construct zero-sum game Γ^0:
  - Maximizer plays all roles in Γ (i.e., $C_M := C$, $\Delta(C_M) = \Delta$)
  - Minimizer wants a profitable deviation ($C_m := \bigsqcup_i C_i \times C_i$)
- $\pi \in \text{CE}(\Gamma) \iff u_M^0(\pi, y) \geq 0$ for all minimizer strategies y
- Minimax: such a π exists $\iff$ for all mixed minimizer strategies y there is a $\pi^y \in \Delta(\Gamma)$ such that $u_M^0(\pi^y, y) \geq 0$
- Minimax again: For any y, there is such a $\pi^y \in \Delta^\Pi(\Gamma)$
- In fact $\pi \in \text{CE}(\Gamma) \cap \text{conv}\{\pi^y\}$ □

Exchangeable equilibrium existence

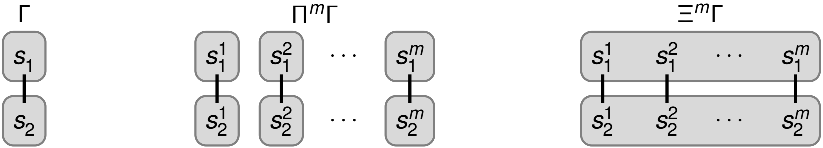
Theorem

For any game with symmetry group G, $\mathrm{XE}_G \neq \emptyset$.

Proof.

- Hart and Schmeidler argument with symmetries added
- G is also a symmetry group of Γ^0
- For each y we can find $\pi^y \in \Delta_G^\Pi(\Gamma)$ s.t. $u_M^0(\pi^y, y) \geq 0$
- Minimax theorem gives CE in $\mathrm{conv}\{\pi^y\} \subseteq \Delta_G^X(\Gamma)$ □

Adding symmetries



Game	Symbol	# players	Symmetries
Original	Γ	n	G
m^{th} power	$\Pi^m\Gamma$	mn	$G \times S_m$ ($G \wr S_m$)
contracted m^{th} power	$\Xi^m\Gamma$	n	$G \times S_m$

Powers of games

- $\Xi^m\Gamma$ has stronger incentive constraints
  - $\mathrm{CE}(\Xi^m\Gamma) \subsetneq \mathrm{CE}(\Pi^m\Gamma)$
- $\Pi^m\Gamma$ has stronger independence constraints
  - $\Delta^{\Pi}_{G\times S_m}(\Pi^m\Gamma) \subsetneq \Delta^{\Pi}_{G\times S_m}(\Xi^m\Gamma)$ (resp. with X in place of Π)

Observations

- $\mathrm{XE}_{G\times S_m}(\Pi^m\Gamma)$ and $\mathrm{XE}_{G\times S_m}(\Xi^m\Gamma)$ are incomparable
- There is a natural map

$$\mathrm{NE}_G(\Gamma) \to \mathrm{XE}_{G\times S_m}(\Pi^m\Gamma) \cap \mathrm{XE}_{G\times S_m}(\Xi^m\Gamma)$$

so we still expect this intersection to be nonempty

Definition

The **order** m **exchangeable equilibria** are

$$\mathrm{XE}_G^m(\Gamma) := \mathrm{XE}_{G\times S_m}(\Pi^m\Gamma) \cap \mathrm{XE}_{G\times S_m}(\Xi^m\Gamma)$$

Higher order exchangeable equilibrium existence

Theorem

For any game with symmetry group G and $m \in \mathbb{N}$, $\mathrm{XE}_G^m \neq \emptyset$.

Proof.

- Similar to XE existence proof □

Theorem

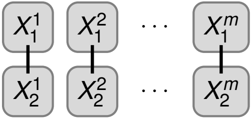
For any game with symmetry group G and $\epsilon > 0$, $\epsilon\text{-}\mathrm{NE}_G \neq \emptyset$.

Proof.

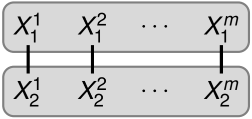
- We will do the symmetric bimatrix case (next slide)
- General case is the same if there is "enough symmetry"
- Otherwise (e.g. arbitrary bimatrix games): symmetrize □

Towards Nash equilibria

$\Pi^m\Gamma$



$\Xi^m\Gamma$



Symmetric bimatrix case (to simplify notation)

- $(X_i^j) \sim \pi \in \mathrm{XE}_{\mathbb{Z}_2}^m$, m large
- X_1^1 is a best reply to $\mathbb{P}(X_2^1 \mid X_1^1, \ldots, X_1^m)$, as is X_1^j
- Random empirical distribution $Y := \frac{1}{m}\sum_{j=1}^m \delta_{X_1^j}$ with values in $\Delta(C_1)$
- Y is a best reply to $\mathbb{P}(X_2^1 \mid X_1^1, \ldots, X_1^m)$
- Exchangeability of X_i^j: $Y \approx \mathbb{P}(X_2^1 \mid X_1^1, \ldots, X_1^m)$
- Y is approximately a best reply to Y with high probability
- $(Y, Y) \in \epsilon\mathrm{NE}_{\mathbb{Z}_2}$ with high probability, $\epsilon \to 0$ as $m \to \infty$

Nash's Theorem

For any game Γ and symmetry group G, $\mathrm{NE}_G \neq \emptyset$.

Proof.

- Sets $\epsilon\,\mathrm{NE}_G$ are nonempty, compact, Hausdorff, nested
- $\mathrm{NE}_G = \bigcap_{\epsilon>0} \epsilon\,\mathrm{NE}_G \neq \emptyset$ □

Concluding remarks

Symmetry

- Theorem still applies for trivial G, so NE $\neq \emptyset$ for all games
- Nonetheless symmetry is fundamental to the argument
- No obvious direct path to NE $\neq \emptyset$ without symmetries

Exchangeable equilibria

- Natural mathematical objects interesting in their own right
- Game theoretic interpretations
- Computable in polynomial time
- To hear more, come to my talk in Brazil!