Games on Manifolds

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Outline

- Previous work
- Background about manifolds
- Game forms
- Potential games
- Equilibria and existence
- Generalizations of games
- Conclusions

Previous work

- Debreu - “Smooth Preferences” (1972)
- Ekeland - “Topologie Différentielle et Théorie des Jeux” (1974)
- Thom - “L’Optimisation Simultanée et la Théorie des Jeux en Topologie Différentielle” (1974)
- Monderer and Shapley - “Potential Games” (1996)

Smooth manifolds

- Topological spaces which look locally like $\mathbb{R}^n$
- Examples: $\mathbb{R}^n$, S^n, $T^2 = S^1 \times S^1$
- Non-examples: $\mathbb{Q}$, $[-1, 1] \subset \mathbb{R}$, $\{(x, y) \in \mathbb{R}^2 | xy = 0\}$
- $C^\infty(M)$ = set of smooth functions $M \to \mathbb{R}$
  - i.e., functions which look locally like smooth functions on $\mathbb{R}^n$

Differentiation

- The first-order behavior of a function on $\mathbb{R}^n$ at a point is given by an element of $\mathbb{R}^n$
- The same is true in a manifold, but there is no natural way to "compare" derivatives at different points $x, y \in M$
- We view these as living in different **cotangent spaces** called $T_x^*(M) \cong \mathbb{R}^n \cong T_y^*(M)$
- These glue together to form the **cotangent bundle** $T^*(M)$, a 2n-dimensional manifold

Differentiation II

- If $f \in C^\infty(M)$ then its derivative df is a smooth map of points $x \in M$ to elements $df(x) \in T_x^*(M)$
- We write $df \in \Omega^1(M)$ ("1-forms")
- Elements of $\Omega^1(M)$ which are of the form df for $f \in C^\infty(M)$ are called **exact**
- If $\omega \in \Omega^1(M)$ looks locally like df for some f (depending on where we look) then we write $d\omega = 0$ and say ω is **closed**
- Example: $M = S^1$, $\omega =$ "$d\theta$" $\in \Omega^1(M)$ is closed but not exact since θ is not a well-defined function

Our setup

- Strategic form game
- Two players for simplicity
- Pure strategy sets are smooth, compact, connected manifolds M, N
- Utility functions are $u, v \in C^\infty(M \times N)$

Game forms

- $T^*_{(m,n)}(M \times N) \cong T^*_m(M) \oplus T^*_n(N)$
- $\psi_M, \psi_N \in C^\infty(T^*(M \times N), T^*(M \times N))$ are the smooth bundle maps which kill the second and first component of this direct sum, resp.
- Define a **game form** $\omega = \psi_M(du) + \psi_N(dv)$
- This encapsulates all strategically relevant information about the game
- That is, $\omega = \tilde{\omega}$ iff $u - \tilde{u}$ is a function of n alone and $v - \tilde{v}$ is a function of m alone

Potential games

- A game is an **(exact) potential game** if it is equivalent under the above def. to a game with $u = v$ [Monderer + Shapley]
- Potential games have (pure) Nash equilibria
- The above shows that a game is an exact potential game if and only if the game form ω is exact
- The Poincaré lemma states that a 1-form on a convex subset of $\mathbb{R}^k$ is exact if and only if it is closed
- $\therefore$ For games with convex strategy sets being a potential game is a local condition $d\omega = 0$ [M+S]

Potential games II

- As mentioned above, in general $\omega = df$ implies $d\omega = 0$ but not conversely
- However, one can show that for game forms the converse is true
- Theorem: For any M and N a game is a potential game iff ω is closed
- Proof: Define a candidate potential function as in Thm. 4.5 of [M+S]. Most of the work is in using technical tools (e.g., Künneth formula, de Rham thm.) to show that this is well-defined.

Solution concepts

- **(Pure) Nash equilibrium**: no player can improve by deviating

$$\Downarrow$$

- **Local (Nash) equilibrium**: no player can improve by deviating within a neighborhood of his strategy

$$\Downarrow$$

- **First order (Nash) equilibrium**: $\omega(m, n) = 0$

Existence of first order equilibria

- Theorem [Ekeland]: If the Euler characteristics $\chi(M)$ and $\chi(N)$ are nonzero then there exists a first-order Nash equilibrium.

- Proof: If not, fix a Riemannian metric on $M \times N$ and use it to map ω to a nonvanishing vector field X on $M \times N$. By the Poincaré-Hopf theorem

$$0 = \sum_{\substack{(m,n): \\ X(m,n)=0}} \text{index}_{(m,n)}(X) = \text{index}(X)$$

$$= \chi(M \times N) = \chi(M)\chi(N).$$

□

Nonexistence of local equilibria

- Theorem: For any strategy manifolds M and N there exists a game with no local equilibria
- Proof: The game with strategy sets $[-1, 1]$ for both players and utilities $u(x, y) = -v(x, y) = (x - y)^2$ has no local equilibria. Using Morse functions we can construct smooth open maps $f : M \to [-1, 1]$ and $g : N \to [-1, 1]$. Then $u(f(m), g(n))$ and $v(f(m), g(n))$ are smooth utilities on $M \times N$ which admit no local equilibria by def. of an open map.

Generalizations of games

- For $m_0 \in M$ define $i_{m_0}(n) = (m_0, n)$ and similarly define j_{n_0} for $n_0 \in N$.
- Theorem: A 1-form ω is a game form if and only if $i^*_{m_0}(\omega) \in \Omega^1(N)$ and $j^*_{n_0}(\omega) \in \Omega^1(M)$ are exact for all $m_0 \in M$ and $n_0 \in N$
- i.e., a game form is one in which each player's preferences come from utility functions
- Define a **local game** to mean the players' utilities look locally like they come from utility functions.
- i.e., replace exact with closed above

Open questions

- Examples of local games
- The games constructed above with no local equilibria are highly degenerate in a well-defined sense. Is there a construction which avoids this or might nondegenerate games on certain manifolds always admit local equilibria?
- Extensions from smooth utilities to C^2 utilities and to manifolds with boundary