Exchangeable Equilibria in Symmetric Bimatrix Games

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Outline

Topics

- Introduction to exchangeable equilibria
- Exchangeability of random variables
- Definition of exchangeable equilibria
- De Finetti's Theorem on exchangeable random variables
- Interpretation, characterization of exchangeable equilibria
- Separation example
- Multiplayer interpretation
- Elementary existence proof

Thought experiment

Setup

- Pick two random Bayesian rational agents off the street
- Put them in separate rooms
- Give them each the table for a symmetric bimatrix game:

$$\begin{bmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{bmatrix}$$

- Tell them this is what you have done
- Ask each what strategy he would play

Main question

- What should we expect to happen?

Main idea

More formal setup

- Population of interchangeable players
- Two play a game with symmetric payoffs
- We are outside observers predicting play
- Environment gives no way to break symmetry

Immediate implications

- Bayesian rationality $\Rightarrow$ play is a correlated equilibrium W
- Interchangeability $\Rightarrow$ $W = W^T$

Our claim

- Not all symmetric correlated equilibria are reasonable
- Some are "more symmetric" than others

Thought experiment, continued

Sneaky trick

- Suppose we pick three people
- Again put each in a room
- Give all the same bimatrix game
- Ask what they would do
- Call their responses X_1, X_2, and X_3

Implications

- Ignoring X_3, X_1 and X_2 should be a correlated equilibrium
- Joint distribution of the X_i invariant under relabeling

Example

Non-example

Game: $\begin{bmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{bmatrix}$ Correlated equilibrium: $\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$

- Is this a reasonable joint distribution for X_1 and X_2?

Claim

No symmetric distribution of X_1, X_2, X_3 has marginal $\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$.

Proof.

- With probability one $X_i \neq X_j$ for all $i \neq j$.
- By the pigeonhole principle, $X_i = X_j$ for some $i \neq j$.

Exchangeability

Definition

A sequence of random variables $X_1, X_2, \ldots$ is **exchangeable** if permuting finitely many of the X_k doesn't affect its distribution.

Properties

i.i.d.

$\Rightarrow$ exchangeable

$\Rightarrow$ X_j, X_k marginal is symmetric, fixed for any $j \neq k$

$\Rightarrow$ identically distributed

Exchangeable but not independent examples

- Distribution of X_1 arbitrary, all $X_k = X_1$ almost surely
- Repeated flips of a coin with a random bias

Exchangeable equilibria

Definition

An **exchangeable equilibrium** is a correlated equilibrium which is extendable to an exchangeable distribution.

Remarks

- Natural limit of thought experiment
- Correlated equilibrium $\Leftrightarrow$ Bayesian rationality
- Exchangeable distribution $\Leftrightarrow$ Bayesian model for interchangeable members of population
- Symmetric Nash equilibria are i.i.d. distributions
- $\text{NE}_{\text{Sym}} \subset \text{XE}_{\text{Sym}} \subset \text{CE}_{\text{Sym}}$

De Finetti's theorem

Theorem (de Finetti)

A sequence $X_1, X_2, \ldots$ is exchangeable if and only if it is i.i.d. conditioned on some random parameter Λ.

Interpretation

- In exchangeable equilibria players react symmetrically to noisy measurement of environment
- If parameter Λ were common knowledge play would be a (random, symmetric) Nash equilibrium
- This corresponds to perfect measurements, but in general exchangeable equilibria measurements may be noisy
- E.g.: Sunspots may or may not occur; if they do players may or may not notice
- Standard game theoretic insight: Players may be better off with less info, i.e., noisier measurements

Completely positive matrices

Definition

Let $Z = \{zz^T \mid z \in \mathbb{R}^{m\times 1} \geq 0\}$ be the set of symmetric, rank 1, nonnegative matrices. The set of **completely positive (CP)** matrices is $\text{conv}(Z)$.

Observation

- The probability matrices in Z (those whose entries sum to 1) are the joint distributions of i.i.d. random variables.

Corollary (of de Finetti's theorem)

The joint distribution of random variables X_1, X_2 is completely positive if and only if it can be extended to an exchangeable sequence $X_1, X_2, \ldots$.

Characterization of exchangeable equilibria

Corollary

The exchangeable equilibria are the correlated equilibria which are completely positive as matrices.

Consequences

- The set of exchangeable equilibria is convex and compact
- $\mathrm{NE_{Sym}} \subset \mathrm{conv}(\mathrm{NE_{Sym}}) \subset \mathrm{XE_{Sym}} \subset \mathrm{CE_{Sym}}$
  - These sets can all be different (example soon)

Sidenote

- Can also use CP matrices to characterize $\mathrm{conv}(\mathrm{NE_{Sym}})$
- Can then prove that $\mathrm{conv}(\mathrm{NE_{Sym}}) = \mathrm{XE_{Sym}}$ for 2×2 games

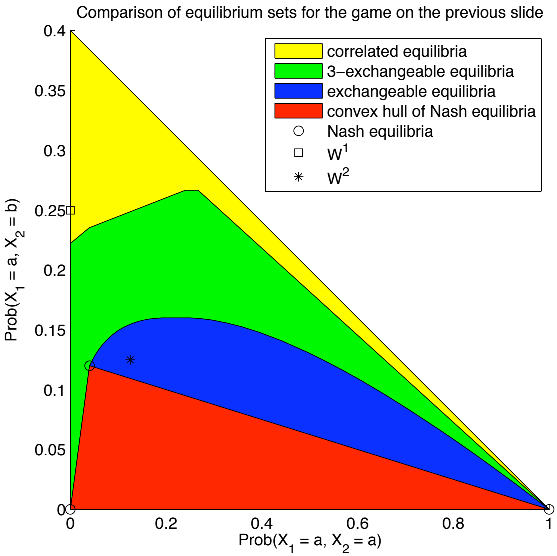
Separation example

Example game

(u_1, u_2)	a	b	c
a	$(5,5)$	$(5,4)$	$(0,0)$
b	$(4,5)$	$(4,4)$	$(4,5)$
c	$(0,0)$	$(5,4)$	$(5,5)$

- Symmetric Nash equilibria:
  $\begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0.2 & 0.6 & 0.2\end{bmatrix}$
- Non-exchangeable correlated equilibrium:
  $$W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \qquad \text{(zero diagonal)}$$
- Exchangeable equilibrium not in $\text{conv}(\text{NE}_{\text{Sym}})$:
  $$W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}^T + \frac{1}{2}\begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^T$$

Separation example, plotted



- Correlated equil. which is not exchangeable:

$$W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

- Exchangeable equil. not in conv(Nash):

$$W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

Existence

Theorem (Nash)

A symmetric bimatrix game has a symmetric Nash equilibrium.

Remarks

- In particular this implies exchangeable equilibria exist
- There are several more elementary proofs
- One is an adaptation of Hart and Schmeidler's proof of existence of correlated equilibria
    - Adding a limiting argument we can prove Nash's theorem itself in full generality
- We give a different proof based on the statement of Hart and Schmeidler's result

Multiplayer interpretation

Definition

The *n*-**player extension** of a symmetric bimatrix game Γ is the n-player game Γ^n in which each pair of players plays Γ and each player's utility is the sum of his utilities from these subgames.

Notation

- Set of strategy profiles: $(C_1)^n = C_1 \times \cdots \times C_1$
- Set of correlated strategies symmetric under permuting the players: $\Delta_{Sym}((C_1)^n)$
- Call the symmetric correlated equilibria $\mathrm{CE}_{\mathrm{Sym}}(\Gamma^n)$
- Marginalization onto first m players: $\mu^n_m : \Delta_{Sym}((C_1)^n) \to \Delta_{Sym}((C_1)^m)$

Multiplayer extension lemma

Lemma

Let $\pi \in \Delta_{Sym}((C_1)^n)$*. Then* $\pi \in \mathrm{CE}_{\mathrm{Sym}}(\Gamma^n)$ *if and only if* $\mu_2^n(\pi) \in \mathrm{CE}_{\mathrm{Sym}}(\Gamma)$*. In particular* $\mu_m^n : \mathrm{CE}_{\mathrm{Sym}}(\Gamma^n) \to \mathrm{CE}_{\mathrm{Sym}}(\Gamma^m)$*.*

Proof.

$$\begin{aligned}
\mathbb{E}_\pi\, u_1^n(f(X_1), X_2, \ldots, X_n) &= \mathbb{E}_\pi \sum_{i=2}^{n} u_1(f(X_1), X_i) \\
&= \sum_{i=2}^{n} \mathbb{E}_\pi\, u_1(f(X_1), X_i) \\
&= \sum_{i=2}^{n} \mathbb{E}_{\mu_2^n(\pi)}\, u_1(f(X_1), X_2) \\
&= (n-1)\, \mathbb{E}_{\mu_2^n(\pi)}\, u_1(f(X_1), X_2) \qquad \square
\end{aligned}$$

Equivalence with original definition

Original definition

- An XE is a CE which extends to an exchangeable distribution.

Alternative definition

- An XE is a CE which extends to $\Delta_{Sym}((C_1)^n)$ for all n.

Corollary

- *An* XE *is a* CE *which extends to* $\mathrm{CE}_{\mathrm{Sym}}(\Gamma^n)$ *for all n.*

Interpretation

- Exchangeable equilibria are symmetric correlated equilibria of large games with many identical interactions

Elementary proof of existence

Theorem

Any symmetric bimatrix game admits an exchangeable equilibrium.

Proof.

- For all n, $\mathrm{CE}(\Gamma^n)$ is compact, convex, nonempty (HS '89)
- Average over permutations of the players: so is $\mathrm{CE}_{\mathrm{Sym}}(\Gamma^n)$
- For $m < n$:
  - $\mu^n_m : \mathrm{CE}_{\mathrm{Sym}}(\Gamma^n) \to \mathrm{CE}_{\mathrm{Sym}}(\Gamma^m)$
  - $\mu^n_2(\mathrm{CE}_{\mathrm{Sym}}(\Gamma^n)) = \mu^m_2(\mu^n_m(\mathrm{CE}_{\mathrm{Sym}}(\Gamma)^n)) \subseteq \mu^m_2(\mathrm{CE}_{\mathrm{Sym}}(\Gamma^m))$
- $\mathrm{XE}(\Gamma) = \bigcap_{n=2}^{\infty} \mu^n_2(\mathrm{CE}_{\mathrm{Sym}}(\Gamma^n))$
- Nested intersection of convex sets is nonempty □

Closing remarks

Interpretations of exchangeable equilibria

- Natural objects between Nash and correlated equilibria
- Right way to maintain symmetry under correlation
- Coordination on noisy measurements of the environment
- Equilibria of games with many simultaneous interactions

Other results

- Extension to multiplayer games / general symmetries
- Can be used to prove Nash's theorem via the separation techniques of HS '89 without fixed point theorems