

Probability Distributions on Partially Ordered Sets and Network Interdiction Games

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Abstract. This article poses the following problem: Does there exist a probability distribution over subsets of a finite partially ordered set (poset), such that a set of constraints involving marginal probabilities of the poset’s elements and maximal chains is satisfied? We present a combinatorial algorithm to positively resolve this question. The algorithm can be implemented in polynomial time in the special case where maximal chain probabilities are affine functions of their elements. This existence problem is relevant for the equilibrium characterization of a generic strategic interdiction game on a capacitated flow network. The game involves a routing entity that sends its flow through the network while facing path transportation costs and an interdictor who simultaneously interdicts one or more edges while facing edge interdiction costs. Using our existence result on posets and strict complementary slackness in linear programming, we show that the Nash equilibria of this game can be fully described using primal and dual solutions of a minimum-cost circulation problem. Our analysis provides a new characterization of the critical components in the interdiction game. It also leads to a polynomial-time approach for equilibrium computation.

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1. Introduction

In this article, we study the problem of showing the existence of a probability distribution over a partially ordered set (or poset) that satisfies a set of constraints involving marginal probabilities of the poset’s elements and maximal chains. This problem is essential for the equilibrium analysis and computation of a generic network interdiction game, in which a strategic interdictor seeks to disrupt the flow of a routing entity. In particular, our existence result on posets enables us to show that the equilibrium structure of the game can be described using primal and dual solutions of a minimum-cost circulation problem.

1.1. Probability Distributions over Posets

For a given finite nonempty poset, we consider a problem in which each element is associated with a value between zero and one; additionally, each maximal chain has a value at most one. We want to determine if there exists a probability distribution over the subsets of the poset such that (i) the probability that each element of the poset is in a subset is *equal to* its corresponding value and (ii) the probability that each maximal chain of the poset intersects with a subset is *as large as* its corresponding value. This problem, denoted (\mathcal{D}), is equivalent to resolving the feasibility of a polyhedral set. However, geometric ideas—such as the ones involving the use of Farkas’ lemma or Carathéodory’s theorem—cannot be applied to solve this problem, because they do not capture the structure of posets. We positively resolve problem (\mathcal{D}) under two conditions that are naturally satisfied for typical situations:

1. The value of each maximal chain is no more than the sum of the values of its elements.
2. The values of the maximal chains satisfy a conservation law: For any decomposition of two intersecting maximal chains, the sum of the corresponding maximal chain values is constant.

Under these two conditions, we prove the feasibility of Problem (\mathcal{D}) (Theorem 1). First, we show that solving (\mathcal{D}) is equivalent to proving that the optimal value of an exponential-size linear optimization problem, denoted (\mathcal{Q}), is no more than one (Proposition 1). Then, to optimally solve (\mathcal{Q}), we design a combinatorial algorithm (Algorithm 1) that exploits the relation between the values associated with the poset's elements and maximal chains. Each iteration of the algorithm involves constructing a subposet, selecting its set of minimal elements, and assigning a specific weight to it. Importantly, in the design of the algorithm, we ensure that the conservation law satisfied by the values associated with the maximal chains of the poset is preserved after each iteration. This design feature enables us to obtain a relation between maximal chains after each iteration, which leads to optimality guarantees of the algorithm (Propositions 2–4). We show that the optimal value of (\mathcal{Q}) is equal to the largest value associated with an element or maximal chain of the poset and is no more than one (Theorem 2).

In the special case where the value of each maximal chain is an affine function of the constituting elements, we refine our combinatorial algorithm to efficiently solve (\mathcal{Q}) (Proposition 5). Our polynomial algorithm (Algorithm 2) relies on subroutines based on the shortest path algorithm in directed acyclic graphs, and does not require the enumeration of maximal chains.

Next, we show that the feasibility of problem (\mathcal{D}) on posets is crucial for the equilibrium analysis of a class of two-player interdiction games on flow networks.

1.2. Network Interdiction Games

We model a network interdiction game between Player 1 (routing entity) that sends its flow through the network while facing heterogeneous path transportation costs and Player 2 (interdictor) who simultaneously chooses an interdiction plan comprised of one or more edges. Player 1 (respectively, Player 2) seeks to maximize the value of effective (respectively, interdicted) flow net the transportation (respectively, interdiction) cost. We adopt mixed strategy Nash equilibria as the solution concept of this game.

Our interdiction game is general in that it models heterogeneous costs of transportation and interdiction. It models the strategic situation in which Player 1 is an operator who wants to route flow (e.g., water, oil, or gas) through pipelines, whereas Player 2 is an attacker who targets multiple pipes in order to steal or disrupt the flow. Another relevant setting is the one where Player 1 is a malicious entity composed of routers who carry illegal (or dangerous) goods through a transportation network (i.e., roads, rivers, etc.) and Player 2 is a security agency that dispatches interdictors to intercept malicious routers and prevent the illegal goods from crossing the network. In both these settings, mixed strategies can be viewed as the players introducing randomization in implementing their respective actions. For instance, Player 1's mixed strategy models a randomized choice of paths for routing its flow of goods through the network, whereas Player 2's mixed strategy indicates a randomized dispatch of interdictors to disrupt or intercept the flow.

The existing literature in network interdiction and robust flow problems has dealt with this type of problems in a sequential (Stackelberg) setting (see Avenhaus and Canty [6], Ball et al. [8], Ratliff et al. [28], Wollmer [32]). Typically, these problems are solved using integer programming techniques and are staple for designing system interdiction and defense (see Aneja et al. [3], Bertsimas et al. [11], Cormican et al. [12], Neumayer et al. [26], Sullivan and Cole Smith [29], Wood [33]). However, these models do not capture the situations in which the interdictor is capable of simultaneously interdicting multiple edges, possibly in a randomized manner. Our model is closely tied to the randomized network interdiction problem considered by Bertsimas et al. [10], in which the interdictor first randomly interdicts a fixed number of edges and then the operator routes a feasible flow in the network. The interdictor's goal is to minimize the largest amount of flow that reaches the destination node. Although this model is equivalent to a simultaneous game, our model differs in that we do not impose any restriction on the number of edges that can be simultaneously interdicted. Additionally, we account for transportation and interdiction costs faced by the players.

Our work is also motivated by previous problems studied in network security games (e.g., Baykal-Gürsoy et al. [9], Gueye et al. [18], Szeto [30]). However, the available results in this line of work are for simpler cases and do not apply to our model. Related to our work are the network security games proposed by Washburn and Wood [31] and Gueye and Marbukh [17]. Washburn and Wood [31] consider a simultaneous game where an evader chooses one source-destination path and the interdictor inspects one edge. The interdictor's (respectively, evader's) objective is to maximize (respectively, minimize) the probability that the evader is detected by the interdictor. Gueye and Marbukh [17] model an operator who routes a feasible flow in the network and an attacker who disrupts one edge. The attacker's (respectively, operator's) goal is to maximize (minimize) the amount of lost flow. The attacker also faces a cost of attack. In contrast, our model allows the interdictor to inspect multiple edges simultaneously, and also accounts for the transportation cost faced by the routing entity.

The generality of our model renders known methods for analyzing security games inapplicable to our game. Indeed, prior work has considered solution approaches based on max-flows and min-cuts, and used these objects as metrics of criticality for network components (see Assadi et al. [4], Dwivedi and Yu [14], Gueye et al. [18]). However, these objects cannot be applied to describe the critical network components in our game because of the heterogeneity of path interdiction probabilities resulting from the transportation costs. A related issue is that computing a Nash equilibrium (NE) of our game is challenging because of the large size of the players' action sets. Indeed, Player 1 (respectively, Player 2) chooses a probability distribution over an infinite number of feasible flows (respectively, exponential number of subsets of edges). Therefore, well-known algorithms for computing (approximate) Nash equilibria are practically inapplicable for this setting (see Gilpin et al. [15], Lipton et al. [23], McMahan et al. [24]). Guo et al. [19] developed a column and constraint generation algorithm to approximately solve their network security game. However, it cannot be applied to our model because of the transportation and interdiction costs that we consider.

Instead, we propose an approach for solving our game based on a minimum-cost circulation problem, which we denote (\mathcal{M}) , and our existence problem on posets (\mathcal{D}) . The main findings are the following:

1. Every Nash equilibrium of the game can be described using primal and dual optimal solutions of (\mathcal{M}) (Theorem 2). Specifically, the expected flow of an equilibrium routing strategy for Player 1 is an optimal flow of (\mathcal{M}) . Furthermore, equilibrium interdiction strategies for Player 2 are such that the marginal interdiction probabilities of the network edges and source-destination paths can be expressed using the optimal dual solutions and the properties of the network. In fact, these equilibrium conditions rely on our results on posets (Theorems 1 and 2) for the existence problem (\mathcal{D}) . The players' payoffs in equilibrium can be expressed in terms of the optimal solutions of (\mathcal{M}) and are independent of the chosen path decomposition of Player 1's strategy. Bertsimas et al. [11] showed that such property does not necessarily hold in path-based formulations of the Robust Maximum Flow Problem (RMFP) with multiple interdictions.

2. Our solution approach shows that Nash equilibria of the game can be computed in polynomial time: The first step consists of solving the minimum-cost circulation problem (\mathcal{M}) using known algorithms (see Karmarkar [22] and Orlin et al. [27]). The optimal flow is shown to be an equilibrium routing strategy for Player 1. Using the optimal dual solution, the second step of our approach consists of running our polynomial algorithm on posets (Algorithm 2) to construct an equilibrium interdiction strategy for Player 2 that satisfies marginal interdiction probabilities. This result contrasts with the NP-hardness of the RMFP (Disser and Matuschke [13]).

3. The critical components in the network can be computed from a primal-dual pair of solutions of (\mathcal{M}) that satisfy strict complementary slackness. Specifically, the primal (respectively, dual) solution provides the paths (respectively, edges) that are chosen (respectively, interdicted) in at least one Nash equilibrium of the game (Proposition 6). This result generalizes the classical min-cut-based metrics of network criticality previously studied in the network interdiction literature (see Assimakopoulos [5], McMasters and Mustin [25], Washburn and Wood [31], Wood [33]). Indeed, we show that in our more general setting, multiple edges in a source-destination path may be interdicted in equilibrium and cannot be represented with a single cut of the network.

The rest of the paper is organized as follows: In Section 2, we pose our existence problem on posets and introduce our main feasibility result. Section 3 presents and analyzes a combinatorial algorithm for solving the existence problem. A polynomial implementation of the algorithm is described in Section 4 when the maximal chain values are affine. Applications of our results on posets are then demonstrated in Section 5, where we study our strategic network interdiction game. Lastly, we provide some concluding remarks in Section 6.

2. Probability Distributions on Posets

In this section, we first recall some standard definitions in order theory. We then pose our problem of proving the existence of probability distributions over partially ordered sets and introduce our main result about its feasibility.

2.1. Preliminaries

A finite *partially ordered set* or *poset* P is a pair (X, \preceq) , where X is a finite set and \preceq is a partial order on X , that is, \preceq is a binary relation on X satisfying the following:

- Reflexivity: For all $x \in X$, $x \preceq x$ in P .
- Antisymmetry: For all $(x, y) \in X^2$, if $x \preceq y$ in P and $y \preceq x$ in P , then $x = y$.
- Transitivity: For all $(x, y, z) \in X^3$, if $x \preceq y$ in P and $y \preceq z$ in P , then $x \preceq z$ in P .

Given $(x, y) \in X^2$, we denote $x \prec y$ in P if $x \preceq y$ in P and $x \neq y$. We say that x and y are *comparable* in P if either $x \prec y$ in P or $y \prec x$ in P . On the other hand, x and y are *incomparable* in P if neither $x \prec y$ in P nor $y \prec x$ in P . We say that x is *covered* in P by y , denoted $x \prec\cdot y$ in P , if $x \prec y$ in P and there does not exist $z \in X$ such that $x \prec z$ in P and $z \prec y$ in P . When there is no confusion regarding the poset, we abbreviate $x \preceq y$ in P by writing $x \preceq y$, etc.

Let Y be a nonempty subset of X , and let $\preceq|_Y$ denote the restriction of \preceq to Y . Then, $\preceq|_Y$ is a partial order on Y and $(Y, \preceq|_Y)$ is a *subposet* of P . A poset $P = (X, \preceq)$ is called a *chain* (respectively, *antichain*) if every distinct pair of elements in X is comparable (respectively, incomparable) in P . Given a poset $P = (X, \preceq)$, a nonempty subset $Y \subseteq X$ is a *chain* (respectively, an *antichain*) in P if the subposet $(Y, \preceq|_Y)$ is a chain (respectively, an antichain). A single element of X is both a chain and an antichain.

Given a poset $P = (X, \preceq)$, an element $x \in X$ is a *minimal* element (respectively, *maximal* element) if there are no elements $y \in X$ such that $y \prec x$ (respectively, $x \prec y$). Note that any chain has a unique minimal and maximal element. A chain $C \subseteq X$ (respectively, antichain $A \subseteq X$) is *maximal* in P if there are no other chains C' (respectively, antichains A') in P that contain C (respectively, A). Let \mathcal{C} and \mathcal{A} respectively denote the set of maximal chains and antichains in P . A maximal chain $C \in \mathcal{C}$ of size n can be represented as $C = \{x_1, \dots, x_n\}$ where for all $k \in \llbracket 1, n-1 \rrbracket$, $x_k \prec x_{k+1}$. We state the following property:

Lemma 1. *Given a finite nonempty poset P , the set of minimal elements of P is an antichain of P and intersects with every maximal chain of P .*

The proof is in Appendix A.

Given a poset $P = (X, \preceq)$, we consider its directed *cover* graph, denoted $H_P = (X, E_P)$. H_P is a directed acyclic graph whose set of nodes is X and whose set of edges is given by $E_P := \{(x, y) \in X^2 \mid x \prec y\}$. When H_P is represented such that for all $(x, y) \in X^2$ with $x \prec y$, the vertical coordinate of the node corresponding to y is higher than the vertical coordinate of the node corresponding to x , the resulting diagram is called a *Hasse diagram* of P .

We now introduce the notion of subposet generated by a subset of maximal chains. Given a poset $P = (X, \preceq)$, let $X' \subseteq X$ be a subset of elements, let $\mathcal{C}' \subseteq \mathcal{C}$ be a subset of maximal chains of P , and consider the binary relation $\preceq_{\mathcal{C}'}$ defined as follows: for all $(x, y) \in X'^2$, $x \preceq_{\mathcal{C}'} y$ if and only if $(x = y)$ or (there exists $C \in \mathcal{C}'$ such that $x, y \in C$ and $x \prec y$). Furthermore, we assume that if $C^1 = \{x_{-k}, \dots, x_{-1}, x^*, x_1, \dots, x_n\}$ and $C^2 = \{y_{-l}, \dots, y_{-1}, x^*, y_1, \dots, y_m\}$ are in \mathcal{C}' and intersect in $x^* \in X'$, then \mathcal{C}' also contains $C_1^2 = \{x_{-k}, \dots, x_{-1}, x^*, y_1, \dots, y_m\}$ and $C_2^1 = \{y_{-l}, \dots, y_{-1}, x^*, x_1, \dots, x_n\}$. In other words, \mathcal{C}' preserves the decomposition of maximal chains intersecting in X' . Then, the following lemma holds:

Lemma 2. *Consider the poset $P = (X, \preceq)$, a subset $X' \subseteq X$, and a subset $\mathcal{C}' \subseteq \mathcal{C}$ that preserves the decomposition of maximal chains intersecting in X' . Then, $P' = (X', \preceq_{\mathcal{C}'})$ is also a poset. Furthermore, for any maximal chain C of P' of size at least two, there exists a maximal chain C' in \mathcal{C}' such that $C = C' \cap X'$.*

The proof is in Appendix A.

The subposet $P' = (X', \preceq_{\mathcal{C}'})$ of P in Lemma 2 satisfies the property that if two elements in X' are comparable in P and belong to a same maximal chain $C \in \mathcal{C}'$, then they are also comparable in P' . Graphically, this is equivalent to removing the edges from the Hasse diagram H_P if their two end nodes do not belong to a same maximal chain $C \in \mathcal{C}'$.

Example 1. Consider the poset P represented by the Hasse diagram H_P in Figure 1.

We observe that $1 \prec 4$, $2 \prec 3$; 1 and 3 are comparable but 4 and 6 are incomparable; $\{2, 4\}$ is a chain in P but is not maximal because it is contained in the maximal chain $\{2, 3, 4\}$. Similarly, $\{4\}$ is an antichain in P but is not maximal because it is contained in the maximal antichain $\{4, 5\}$. The sets of maximal chains and antichains of P are given by $\mathcal{C} = \{\{1, 3, 4\}, \{2, 3, 5, 6\}, \{1, 3, 5, 6\}, \{2, 3, 4\}\}$ and $\mathcal{A} = \{\{1, 2\}, \{3\}, \{4, 5\}, \{4, 6\}\}$, respectively. The set of minimal elements of P is given by $\{1, 2\}$ and intersects with every maximal chain in \mathcal{C} . Finally, $P' = (X', \preceq_{\mathcal{C}'})$, where $X' = \{1, 2, 3, 4, 6\}$ and $\mathcal{C}' = \{\{1, 3, 5, 6\}, \{2, 3, 5, 6\}\}$ is a poset as illustrated in Figure 1. \triangle

Figure 1. On the left is a Hasse diagram of a poset P . On the right is a Hasse diagram of the subposet $P' = (X', \preceq_{\mathcal{C}'})$ of P , where $X' = \{1, 2, 3, 4, 6\}$ and $\mathcal{C}' = \{\{1, 3, 5, 6\}, \{2, 3, 5, 6\}\}$.



2.2. Problem Formulation and Main Result

Consider a finite nonempty poset $P = (X, \preceq)$. Let $\mathcal{P} := 2^X$ denote the power set of X , and let $\Delta(\mathcal{P}) := \{\sigma \in \mathbb{R}_{\geq 0}^{\mathcal{P}} \mid \sum_{S \in \mathcal{P}} \sigma_S = 1\}$ denote the set of probability distributions over \mathcal{P} . We are concerned with the setting where each element $x \in X$ is associated with a value $\rho_x \in [0, 1]$, and each maximal chain $C \in \mathcal{C}$ has a value $\pi_C \leq 1$. Our problem is to determine if there exists a probability distribution $\sigma \in \Delta(\mathcal{P})$ such that for every element $x \in X$, the probability that x is in a subset $S \in \mathcal{P}$ is equal to ρ_x ; and for every maximal chain $C \in \mathcal{C}$, the probability that C intersects with a subset $S \in \mathcal{P}$ is at least π_C . That is,

$$\begin{aligned}
 (\mathcal{D}): \quad \exists \sigma \in \mathbb{R}_{\geq 0}^{\mathcal{P}} \text{ such that } & \begin{cases} \sum_{\{S \in \mathcal{P} \mid x \in S\}} \sigma_S = \rho_x, & \forall x \in X, & (1a) \\ \sum_{\{S \in \mathcal{P} \mid S \cap C \neq \emptyset\}} \sigma_S \geq \pi_C, & \forall C \in \mathcal{C}, & (1b) \\ \sum_{S \in \mathcal{P}} \sigma_S = 1. & & (1c) \end{cases}
 \end{aligned}$$

For the case in which $\pi_C \leq 0$ for all maximal chains $C \in \mathcal{C}$, Constraints (1b) can be removed and the feasibility of (D) follows from Carathéodory’s theorem. However, no known results can be applied to the general case. Note that although (1a)–(1c) form a polyhedral set, Farkas’ lemma cannot be directly used to evaluate its feasibility. Instead, in this article, we study the feasibility of (D) using order-theoretic properties of the problem. We assume two natural conditions on $\rho = (\rho_x)_{x \in X}$ and $\pi = (\pi_C)_{C \in \mathcal{C}}$, which we introduce next.

Firstly, for feasibility of (D), ρ and π must necessarily satisfy the following inequality:

$$\forall C \in \mathcal{C}, \quad \sum_{x \in C} \rho_x \geq \pi_C. \tag{2}$$

Indeed, if (D) is feasible, then for $\sigma \in \mathbb{R}_{\geq 0}^{\mathcal{P}}$ satisfying (1a)–(1c), the following holds:

$$\forall C \in \mathcal{C}, \quad \sum_{x \in C} \rho_x \stackrel{(1a)}{=} \sum_{x \in C} \sum_{\{S \in \mathcal{P} \mid x \in S\}} \sigma_S = \sum_{S \in \mathcal{P}} \sigma_S \sum_{x \in C} \mathbb{1}_{\{x \in S\}} = \sum_{S \in \mathcal{P}} \sigma_S |S \cap C| \stackrel{(1b)}{\geq} \sum_{\{S \in \mathcal{P} \mid S \cap C \neq \emptyset\}} \sigma_S \geq \pi_C.$$

That is, the necessity of (2) follows from the fact that for any probability distribution over \mathcal{P} , and any subset of elements $C \subseteq X$, the probability that C intersects with a subset $S \in \mathcal{P}$ is upper bounded by the sum of the probabilities with which each element in C is in a subset $S \in \mathcal{P}$.

Secondly, we assume that π satisfies a specific condition for each pair of maximal chains that intersect each other. Consider any pair of maximal chains C^1 and C^2 of P , with $C^1 \cap C^2 \neq \emptyset$. Let $x^* \in C^1 \cap C^2$, and let us rewrite $C^1 = \{x_{-k}, \dots, x_{-1}, x^*, x_1, \dots, x_n\}$ and $C^2 = \{y_{-l}, \dots, y_{-1}, x^*, y_1, \dots, y_m\}$. Then, P also contains two maximal chains $C_1^1 = \{x_{-k}, \dots, x_{-1}, x^*, y_1, \dots, y_m\}$ and $C_2^1 = \{y_{-l}, \dots, y_{-1}, x^*, x_1, \dots, x_n\}$ that satisfy $C^1 \cup C^2 = C_1^1 \cup C_2^1$; see Figure 2 for an illustration. We require π to satisfy the following condition:

$$\pi_{C^1} + \pi_{C^2} = \pi_{C_1^1} + \pi_{C_2^1}. \tag{3}$$

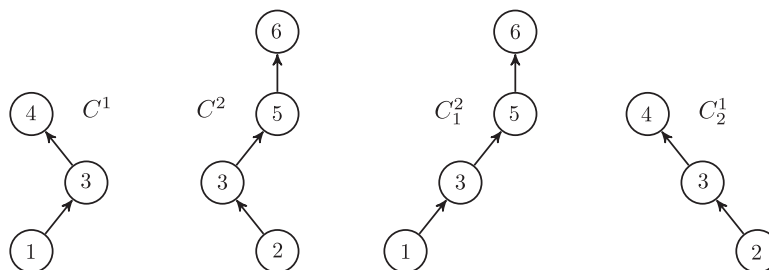
Essentially, (3) can be viewed as a *conservation law* on the maximal chains in \mathcal{C} .

We now present our main result regarding the feasibility of (D), under Conditions (2) and (3).

Theorem 1. *The problem (D) is feasible for any finite nonempty poset (X, \preceq) , with parameters $\rho = (\rho_x) \in [0, 1]^X$ and $\pi = (\pi_C) \in (-\infty, 1]^{\mathcal{C}}$ that satisfy (2) and (3).*

This result plays a crucial role in solving a two-player interdiction game on a flow network (Section 5). The game involves a “router” who sends a flow of goods to maximize his or her value of flow crossing the network while facing transportation costs and an “interdictor” who inspects one or more network edges to maximize the

Figure 2. Four maximal chains of the poset shown in Figure 1.



value of interdicted flow while facing interdiction costs. Our equilibrium analysis in Section 5 shows that interdiction strategies in Nash equilibria interdict each edge x with a probability ρ_x , and interdict each path C with a probability at least π_C . Essentially, for this game, (ρ_x) and (π_C) are governed by network properties, such as edge transportation and interdiction costs, and naturally satisfy (2) and (3). When the network is a directed acyclic graph, a partial order can be defined on the set of edges, such that the set of maximal chains is exactly the set of source-destination paths of the network. Thus, showing the existence of interdiction strategies satisfying the abovementioned equilibrium conditions is an instantiation of the problem (D). In fact, Theorem 1 is useful for deriving several properties satisfied by the equilibrium strategies of this network interdiction game.

It is important to note that (D) may not be feasible if the conservation law (3) is not satisfied, as illustrated in the following counterexample:

Example 2. Let P be the poset represented by the Hasse diagram in Figure 3.

In this poset, the maximal chains are $C^1 = \{1, 3, 5\}$, $C^2 = \{1, 4, 5\}$, $C^3 = \{1, 4, 6\}$, $C^4 = \{2, 4, 5\}$, $C^5 = \{2, 4, 6\}$. Consider the following values: $\rho_x = 0.4$ for $x \in \{1, 4, 5\}$ and $\rho_x = 0$ for $x \in \{2, 3, 6\}$; $\pi_{C^5} = 0.4$ and $\pi_C = 0.8$ for $C \in \mathcal{C} \setminus \{C^5\}$. We note that (2) is satisfied. However, $\pi_{C^2} + \pi_{C^5} = 1.2 \neq 1.6 = \pi_{C^3} + \pi_{C^4}$, which violates (3). If $\sigma \in \mathbb{R}_{\geq 0}^{\mathcal{P}}$ satisfies (1a) and (1b), then, necessarily, $\sigma_{\{x\}} = 0.4$ for all $x \in \{1, 4, 5\}$, which violates (1c). Thus, problem (D) is infeasible for this example. \triangle

Next, we show that (D) is feasible if and only if the optimal value of a linear program is no more than one.

2.3. Equivalent Optimization Problem

We observe that when $\sum_{x \in X} \rho_x \leq 1$, a trivial solution for (D) is given by $\tilde{\sigma}_{\{x\}} = \rho_x$ for all $x \in X$, and $\tilde{\sigma}_\emptyset = 1 - \sum_{x \in X} \rho_x$. The vector $\tilde{\sigma}$ so constructed indeed represents a probability distribution over \mathcal{P} and satisfies Constraints (1a). Furthermore, for each maximal chain $C \in \mathcal{C}$, $\sum_{\{S \in \mathcal{P} \mid S \cap C \neq \emptyset\}} \tilde{\sigma}_S = \sum_{x \in C} \rho_x \stackrel{(2)}{\geq} \pi_C$. Therefore, $\tilde{\sigma}$ is a feasible solution of (D). However, in general, $\sum_{x \in X} \rho_x$ may be larger than one, which prevents the aforementioned construction of $\tilde{\sigma}$ from being a probability distribution. Thus, to construct a feasible solution of (D), some probability must be assigned to subsets of elements of size larger than one. This is governed by the following quantity, defined for each maximal chain $C \in \mathcal{C}$:

$$\delta_C := \sum_{x \in C} \rho_x - \pi_C. \tag{4}$$

The role of $\delta = (\delta_C)_{C \in \mathcal{C}}$ in assigning probabilities to subsets of elements can be better understood by considering the following optimization problem:

$$(Q): \text{ minimize } \sum_{S \in \mathcal{P}} \sigma_S$$

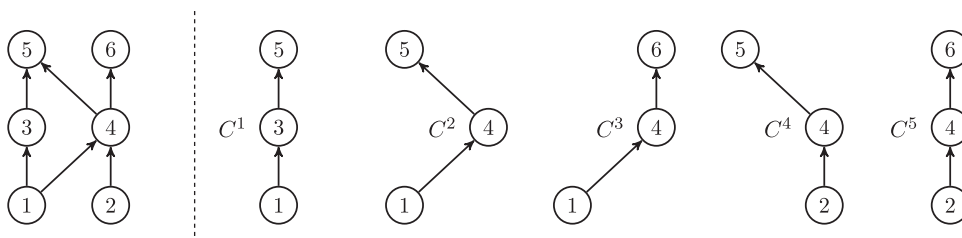
$$\text{subject to } \sum_{\{S \in \mathcal{P} \mid x \in S\}} \sigma_S = \rho_x, \quad \forall x \in X \tag{5}$$

$$\sum_{\{S \in \mathcal{P} \mid |S \cap C| \geq 2\}} \sigma_S (|S \cap C| - 1) \leq \delta_C, \quad \forall C \in \mathcal{C} \tag{6}$$

$$\sigma_S \geq 0, \quad \forall S \in \mathcal{P}.$$

Problems (Q) and (D) are related in that the set of Constraints (1a)–(1b) is equivalent to the set of Constraints (5)–(6); see the proof of Proposition 1. Furthermore, the objective function in (Q) is analogous to Constraint (1c)

Figure 3. Hasse diagram of a poset P (left) and its five maximal chains (right).



in (\mathcal{D}) . The feasibility of (\mathcal{Q}) is straightforward (for example, $\tilde{\sigma}$ constructed above is a feasible solution); however, a feasible solution of (\mathcal{Q}) may not be a probability distribution.

Given a maximal chain $C \in \mathcal{C}$, Constraint (6) bounds the total amount of probability that can be assigned to subsets that contain more than one element in C . One can see that for a subset $S \in \mathcal{P}$ such that $|S \cap C| \leq 1$, the probability σ_S assigned to S does not influence Constraint (6). However, the more elements from C a subset S contains, the smaller the probability that can be assigned to S , because of scaling by the factor $(|S \cap C| - 1)$. Thus, δ determines the amount of probability that can be assigned to larger subsets.

Let $z_{(\mathcal{Q})}^*$ denote the optimal value of (\mathcal{Q}) . Then, the following result holds:

Proposition 1. (\mathcal{D}) is feasible if and only if $z_{(\mathcal{Q})}^* \leq 1$.

Proof of Proposition 1. First, let us show that the set of Constraints (1a)–(1b) is equivalent to the set of Constraints (5)–(6). Let $\sigma \in \mathbb{R}_{\geq 0}^{\mathcal{P}}$ that satisfies $\sum_{\{S \in \mathcal{P} | x \in S\}} \sigma_S = \rho_x$ for all $x \in X$. For every maximal chain $C \in \mathcal{C}$, the following equality holds:

$$\sum_{x \in C} \rho_x = \sum_{x \in C} \sum_{\{S \in \mathcal{P} | x \in S\}} \sigma_S = \sum_{S \in \mathcal{P}} \sigma_S \sum_{x \in C} \mathbb{1}_{\{x \in S\}} = \sum_{\{S \in \mathcal{P} | S \cap C \neq \emptyset\}} \sigma_S |S \cap C|. \tag{7}$$

Therefore, for every maximal chain $C \in \mathcal{C}$, the following equivalence is satisfied:

$$\sum_{\{S \in \mathcal{P} | S \cap C \neq \emptyset\}} \sigma_S \geq \pi_C \stackrel{(4),(7)}{\iff} \delta_C \geq \sum_{\{S \in \mathcal{P} | S \cap C \neq \emptyset\}} \sigma_S (|S \cap C| - 1) = \sum_{\{S \in \mathcal{P} | |S \cap C| \geq 2\}} \sigma_S (|S \cap C| - 1). \tag{8}$$

Now, let us show that (\mathcal{D}) is feasible if and only if the optimal value of (\mathcal{Q}) satisfies $z_{(\mathcal{Q})}^* \leq 1$.

– If there exists $\sigma \in \mathbb{R}_{\geq 0}^{\mathcal{P}}$ that satisfies (1a)–(1b), then we showed that σ is a feasible solution of (\mathcal{Q}) . Furthermore, the objective value of σ is equal to one, which implies that $z_{(\mathcal{Q})}^* \leq 1$.

– If $z_{(\mathcal{Q})}^* \leq 1$, let σ^* be an optimal solution of (\mathcal{Q}) . Necessarily, $\sigma_\emptyset^* = 0$, and the vector $\hat{\sigma} \in \mathbb{R}^{\mathcal{P}}$ defined as follows is feasible for (\mathcal{D}) : $\hat{\sigma}_S = \sigma_S^*$, for every $S \in \mathcal{P} \setminus \emptyset$, and $\hat{\sigma}_\emptyset = 1 - z_{(\mathcal{Q})}^*$. \square

Therefore, proving Theorem 1 is equivalent to showing that $z_{(\mathcal{Q})}^* \leq 1$. In fact, we show a stronger result, which will be crucial for solving our network interdiction game in Section 5:

Theorem 2. $z_{(\mathcal{Q})}^* = \max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$.

It is easy to see that $z_{(\mathcal{Q})}^* \geq \max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$. Indeed, any feasible solution $\sigma \in \mathbb{R}_{\geq 0}^{\mathcal{P}}$ of (\mathcal{Q}) satisfies $\sum_{S \in \mathcal{P}} \sigma_S \geq \sum_{\{S \in \mathcal{P} | x \in S\}} \sigma_S = \rho_x$ for every $x \in X$, and $\sum_{S \in \mathcal{P}} \sigma_S \geq \sum_{\{S \in \mathcal{P} | S \cap C \neq \emptyset\}} \sigma_S \stackrel{(8)}{\geq} \pi_C$ for every $C \in \mathcal{C}$. To show the reverse inequality, we must prove that there exists a feasible solution of (\mathcal{Q}) with objective value equal to $\max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$. This is the focus of the next section.

3. Constructive Proof of Theorem 2

Essentially, we design a combinatorial algorithm to compute a feasible solution of (\mathcal{Q}) with objective value exactly equal to $\max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$. Recall from Section 2.3 that such a feasible solution is optimal for (\mathcal{Q}) and can be used to construct a feasible solution of (\mathcal{D}) ; see the proof of Proposition 1.

Before formally introducing our algorithm, we discuss the main ideas behind its design. In each iteration, the algorithm selects a subset of elements and assigns a positive weight to it. Let us discuss the execution of the first iteration of the algorithm.

Firstly, we determine the collection of subsets that can be assigned a positive weight without violating any of the constraints in the Problem (\mathcal{Q}) . Essentially, this is dictated by the maximal chains $C \in \mathcal{C}$ for which $\delta_C = 0$. Indeed, for any $C \in \mathcal{C}$ with $\delta_C = 0$, the following equivalence holds: $\sum_{\{S \in \mathcal{P} | |S \cap C| \geq 2\}} \underbrace{\sigma_S}_{\geq 0} \underbrace{(|S \cap C| - 1)}_{> 0} \leq 0$ if and only if $\sigma_S = 0$ for all $S \in \mathcal{P}$ such that $|S \cap C| \geq 2$. Therefore, our algorithm must select a subset of elements $S \in \mathcal{P}$ that intersects every maximal chain $C \in \mathcal{C}$ for which $\delta_C = 0$ in at most one element.

To precisely characterize this collection of subsets, we consider the notion of subposet generated by a subset of maximal chains, introduced in Section 2.1. In particular, by considering \mathcal{C}' the set of maximal chains $C \in \mathcal{C}$ such that $\delta_C = 0$, and X' the subset of elements $x \in X$ such that $\rho_x > 0$, we can show (in Proposition 2) that the condition stated in Lemma 2 is satisfied and $P' = (X', \preceq_{\mathcal{C}'})$ is a poset. Interestingly, the subsets of elements that the algorithm can select from at that iteration are the antichains of P' . In any poset, a chain and an antichain intersect in at most one element. By definition of $\preceq_{\mathcal{C}'}$, this implies that $|S \cap C| \leq 1$ for every antichain $S \subseteq X'$ of P' and every maximal chain $C \in \mathcal{C}$ of P such that $\delta_C = 0$.

Now, we need to determine which antichain of P' to select. Let $S' \subseteq X'$ denote the subset of elements selected by the algorithm in the first iteration. Recall that an optimal solution of (Q) satisfies Constraints (1a)–(1b) with the least total amount of weight assigned to subsets of elements of X . Thus, it is desirable that the weight assigned to S' in this iteration contribute toward satisfying all Constraints (1b). To capture this requirement, our algorithm selects S' as the set of minimal elements of P' . The selected S' is an antichain of P' , intersects with every maximal chain of P , and enables us to prove the optimality of the algorithm.

Secondly, we discuss how to determine the maximum amount of weight w' that can be assigned to S' in the first iteration, without violating any of the Constraints (5) and (6). This is governed by the remaining maximal chains $C \in \mathcal{C}$ for which $\delta_C > 0$ and the elements constituting S' . If w' is larger than $\frac{\delta_C}{|S' \cap C| - 1}$ for $C \in \mathcal{C}$ such that $|S' \cap C| \geq 2$, then the corresponding Constraint (6) will be violated. Similarly, w' cannot be larger than any ρ_x , $x \in S'$. Thus, the weight to assign to S' is

$$w' = \min \left\{ \min \{ \rho_x, x \in S' \}, \min \left\{ \frac{\delta_C}{|S' \cap C| - 1}, C \in \mathcal{C} \mid \delta_C > 0 \text{ and } |S' \cap C| \geq 2 \right\} \right\}.$$

At the end of the iteration, the algorithm updates the vectors ρ and δ , as well as the sets of elements X' and maximal chains \mathcal{C}' to consider in subsequent iterations. In particular, we will show that some maximal chains need to be removed in order to preserve the conservation law at each iteration. The algorithm terminates when there are no more elements $x \in X$ with positive ρ_x . We are now in the position to formally present Algorithm 1.

Algorithm 1 (Optimal Solution of (Q))

Input: Finite nonempty poset $P = (X, \preceq)$, and vectors $\rho \in \mathbb{R}_{\geq 0}^X$, $\delta \in \mathbb{R}_{\geq 0}^{\mathcal{C}}$.

Output: Vector $\sigma \in \mathbb{R}_{\geq 0}^P$.

- A1: $\mathcal{C}^1 \leftarrow \mathcal{C}, \quad \rho_x^1 \leftarrow \rho_x, \quad \forall x \in X, \quad \delta_C^1 \leftarrow \delta_C, \quad \forall C \in \mathcal{C}^1$
- A2: $X^1 \leftarrow \{x \in X \mid \rho_x^1 > 0\}, \quad \bar{\mathcal{C}}^1 \leftarrow \{C \in \mathcal{C}^1 \mid \delta_C^1 = 0\}, \quad \hat{\mathcal{C}}^1 \leftarrow \{C \in \mathcal{C}^1 \mid \delta_C^1 > 0\}$
- A3: $k \leftarrow 1$
- A4: **while** $X^k \neq \emptyset$ **do**
- A5: Construct the poset $P^k = (X^k, \preceq_{\bar{\mathcal{C}}^k})$
- A6: Select S^k the set of minimal elements of P^k
- A7: $w^k \leftarrow \min \{ \min \{ \rho_x^k, x \in S^k \}, \min \{ \frac{\delta_C^k}{|S^k \cap C| - 1}, C \in \hat{\mathcal{C}}^k \mid |S^k \cap C| \geq 2 \} \}, \quad \text{and } \sigma_{S^k} \leftarrow w^k$
- A8: $\rho_x^{k+1} \leftarrow \rho_x^k - w^k \mathbb{1}_{\{x \in S^k\}}, \quad \forall x \in X, \quad \text{and } \delta_C^{k+1} \leftarrow \delta_C^k - w^k (|S^k \cap C| - 1) \mathbb{1}_{\{|S^k \cap C| \geq 2\}}, \quad \forall C \in \mathcal{C}$
- A9: $\mathcal{C}^{k+1} \leftarrow \{C \in \mathcal{C}^k \mid \text{the minimal element of } C \cap X^k \text{ in } P \text{ is in } S^k\}$
- A10: $X^{k+1} \leftarrow \{x \in X^k \mid \rho_x^{k+1} > 0\}, \quad \bar{\mathcal{C}}^{k+1} \leftarrow \{C \in \mathcal{C}^{k+1} \mid \delta_C^{k+1} = 0\}, \quad \hat{\mathcal{C}}^{k+1} \leftarrow \{C \in \mathcal{C}^{k+1} \mid \delta_C^{k+1} > 0\}$
- A11: $k \leftarrow k + 1$
- A12: **end while**

We illustrate Algorithm 1 with an example in Appendix B.

Let n^* denote the number of iterations of Algorithm 1. Since it has not yet been shown to terminate, we suppose that $n^* \in \mathbb{N} \cup \{+\infty\}$. For every maximal chain $C \in \mathcal{C}$, let us define the sequence $(\pi_C^k)_{k \in \llbracket 1, n^* + 1 \rrbracket}$ induced by Algorithm 1 as follows:

$$\pi_C^1 := \pi_C, \text{ and for every } k \in \llbracket 1, n^* \rrbracket, \pi_C^{k+1} := \pi_C^k - w^k \mathbb{1}_{\{S^k \cap C \neq \emptyset\}}. \tag{9}$$

Given $k \in \llbracket 1, n^* + 1 \rrbracket$, π_C^k (respectively, ρ_x^k) represents the remaining value associated with the maximal chain $C \in \mathcal{C}$ (respectively, the element $x \in X$) after the first $k - 1$ iterations of the algorithm. For convenience, we let $X^0 \leftarrow X$.

We now proceed with proving Theorem 2. Our proof consists of three main parts:

Part 1: Algorithm 1 is well defined (Proposition 2).

Part 2: It terminates and outputs a feasible solution of (Q) (Proposition 3).

Part 3: It assigns a total weight $\sum_{k=1}^{n^*} w^k$ equal to $\max \{ \max \{ \rho_x, x \in X \}, \max \{ \pi_C, C \in \mathcal{C} \} \}$ at termination (Proposition 4).

3.1. Part 1: Well Definedness of Algorithm 1

To show that Algorithm 1 is well defined, we need to ensure that at each iteration $k \in \llbracket 1, n^* \rrbracket$ of the algorithm, P^k is a poset. Lemma 2 can be applied to show this, provided that we are able to prove that \bar{C}^k preserves the decomposition of maximal chains intersecting in X^k . This property, and some associated results, are stated below:

Proposition 2. *Each iteration of Algorithm 1 is well defined. In particular, for every $k \in \llbracket 1, n^* + 1 \rrbracket$, the following hold:*

i. *For every maximal chain $C \in \mathcal{C}$, δ_C^k determines the remaining weight that can be assigned to subsets that intersect C at more than one element:*

$$\forall C \in \mathcal{C}, \quad \delta_C^k = \sum_{x \in C} \rho_x^k - \pi_C^k, \tag{10}$$

$$\forall C \in \mathcal{C}^k, \quad \delta_C^k \geq 0. \tag{11}$$

ii. C^k preserves the decomposition of maximal chains intersecting in X^{k-1} :

$$\forall (C^1, C^2) \in \mathcal{C}^2 \mid C^1 \cap C^2 \cap X^{k-1} \neq \emptyset, \quad (C^1, C^2) \in (\mathcal{C}^k)^2 \Rightarrow (C_1^2, C_2^1) \in (\mathcal{C}^k)^2.$$

iii. π^k satisfies the conservation law on the maximal chains of \mathcal{C}^k that intersect in X^{k-1} :

$$\forall (C^1, C^2) \in (\mathcal{C}^k)^2 \mid C^1 \cap C^2 \cap X^{k-1} \neq \emptyset, \quad \pi_{C^1}^k + \pi_{C^2}^k = \pi_{C_1^2}^k + \pi_{C_2^1}^k. \tag{12}$$

iv. $P^k = (X^k, \preceq_{\bar{C}^k})$ is a poset.

Proof of Proposition 2. We show (i)–(iv) by induction.

First, consider $k = 1$. Since $\mathcal{C}^1 = \mathcal{C}$, $\rho^1 = \rho$, $\pi^1 = \pi$, and $\delta^1 = \delta$, then (i) follows from (2) and (4). Since $X^0 = X$ and $\mathcal{C}^1 = \mathcal{C}$, then (ii) is automatically satisfied. (iii) is a direct consequence of (3).

Now we apply Lemma 2 to show (iv), that is, $P^1 = (X^1, \preceq_{\bar{C}^1})$ is a poset. Specifically, we show that \bar{C}^1 preserves the decomposition of maximal chains intersecting in X^1 . Consider $C^1, C^2 \in \bar{\mathcal{C}}^1$ such that $C^1 \cap C^2 \cap X^1 \neq \emptyset$. Let $x^* \in C^1 \cap C^2 \cap X^1$, and let us consider the other two maximal chains C_1^2 and C_2^1 , which we know from (ii) are in \mathcal{C}^1 because $X^1 \subseteq X^0$. We can rewrite $C^1 = \{x_{-k}, \dots, x_{-1}, x_0 = x^*, x_1, \dots, x_n\}$ and $C^2 = \{y_{-l}, \dots, y_{-1}, y_0 = x^*, y_1, \dots, y_m\}$. Then, $C_1^2 = \{x_{-k}, \dots, x_{-1}, x^*, y_1, \dots, y_m\}$ and $C_2^1 = \{y_{-l}, \dots, y_{-1}, x^*, x_1, \dots, x_n\}$. From (i)–(iii), since $C^1, C^2 \in \bar{\mathcal{C}}^1$; the conservation law is satisfied by π^1 on the maximal chains in \mathcal{C}^1 intersecting in X^0 ; $C_1^2, C_2^1 \in \mathcal{C}^1$; and since $\delta^1 \geq 0$ on \mathcal{C}^1 :

$$\sum_{i=-k}^n \rho_{x_i}^1 + \sum_{j=-l}^m \rho_{y_j}^1 = \pi_{C^1}^1 + \pi_{C^2}^1 = \pi_{C_1^2}^1 + \pi_{C_2^1}^1 = \sum_{x \in C_1^2} \rho_x^1 + \sum_{x \in C_2^1} \rho_x^1 - \delta_{C_1^2}^1 - \delta_{C_2^1}^1 \leq \sum_{i=-k}^n \rho_{x_i}^1 + \sum_{j=-l}^m \rho_{y_j}^1.$$

Therefore, $\delta_{C_1^2}^1 = \delta_{C_2^1}^1 = 0$ and $C_1^2, C_2^1 \in \bar{\mathcal{C}}^1$. From Lemma 2, $P^1 = (X^1, \preceq_{\bar{C}^1})$ is a poset.

We now assume that (i)–(iv) hold for $k \in \llbracket 1, n^* \rrbracket$ and show that they also hold for $k + 1$:

i. Since P^k is a poset, the k -th iteration of the algorithm is well defined; we can consider the set S^k and the weight w^k at that iteration. Then, for every $C \in \mathcal{C}$, (A8) and (9) give us

$$\sum_{x \in C} \rho_x^{k+1} - \pi_C^{k+1} = \sum_{x \in C} \rho_x^k - \pi_C^k - w^k |S^k \cap C| + w^k \mathbb{1}_{\{S^k \cap C \neq \emptyset\}} = \delta_C^k - w^k (|S^k \cap C| - 1) \mathbb{1}_{\{S^k \cap C \neq \emptyset\}} = \delta_C^{k+1}.$$

Now, consider a maximal chain $C \in \mathcal{C}^k$. Since $\delta_C^k \geq 0$ on \mathcal{C}^k , then $\mathcal{C}^k = \bar{\mathcal{C}}^k \cup \hat{\mathcal{C}}^k$ (from (A10)).

a. If $C \in \bar{\mathcal{C}}^k$, then by definition of $\preceq_{\bar{C}^k}$, $C \cap X^k$ is a chain in P^k . From Lemma 1, S^k is an antichain of P^k . Therefore, $|S^k \cap (C \cap X^k)| \leq 1$. Because $S^k \subseteq X^k$, we obtain that $|S^k \cap C| = |S^k \cap X^k \cap C| \leq 1$. Thus, $\delta_C^{k+1} \stackrel{(A8)}{=} \delta_C^k - w^k (|S^k \cap C| - 1) \mathbb{1}_{\{|S^k \cap C| \geq 2\}} = \delta_C^k = 0$.

b. If $C \in \hat{\mathcal{C}}^k$, then by definition of w^k : $\delta_C^{k+1} \stackrel{(A8)}{=} \delta_C^k - w^k (|S^k \cap C| - 1) \mathbb{1}_{\{|S^k \cap C| \geq 2\}} \stackrel{(A7)}{\geq} 0$.

In summary, for all $C \in \mathcal{C}^k$, $\delta_C^{k+1} \geq 0$. Since $\mathcal{C}^{k+1} \subseteq \mathcal{C}^k$, then for all $C \in \mathcal{C}^{k+1}$, $\delta_C^{k+1} \geq 0$.

ii. Consider $C^1, C^2 \in \mathcal{C}^{k+1} \subseteq \mathcal{C}^k$ such that $C^1 \cap C^2 \cap X^k \neq \emptyset$, and let C_1^2 and C_2^1 be two other maximal chains such that $C_1^2 \cup C_2^1 = C^1 \cup C^2$. Since $X^k \stackrel{(A10)}{\subseteq} X^{k-1}$, then $C^1 \cap C^2 \cap X^{k-1} \neq \emptyset$. Therefore, by inductive hypothesis, $C_1^2, C_2^1 \in \mathcal{C}^k$ as well. Let x_1 (respectively, y_1) denote the minimal element of the chain $C^1 \cap X^k$ (respectively, $C^2 \cap X^k$) in P . Since $C^1, C^2 \in \mathcal{C}^{k+1}$, then $(x_1, y_1) \stackrel{(A9)}{\in} (S^k)^2$. Let $x^* \in X^k$ denote an intersecting element of C^1 and C^2 . Since $C^1 \cap X^k$ is a chain in P , contains x^* , and whose minimal element is x_1 , then necessarily $x_1 \preceq x^*$. Similarly, we obtain that $y_1 \preceq x^*$.

Therefore, the minimal element of $C_1^2 \cap X^k$ (respectively, $C_2^1 \cap X^k$) in P is $x_1 \in S^k$ (respectively, $y_1 \in S^k$). Thus, $C_1^2, C_2^1 \in \mathcal{C}^{k+1}$, and \mathcal{C}^{k+1} preserves the decomposition of maximal chains of P intersecting in X^k .

iii. Now, given C^1, C^2 in \mathcal{C}^{k+1} that intersect in X^k , we just proved that C_1^2 and C_2^1 are in \mathcal{C}^{k+1} as well. Therefore, for all $C \in \{C^1, C^2, C_1^2, C_2^1\}$, $\pi_C^{k+1} \stackrel{(9)}{=} \pi_C^k - w^k$ (because $S^k \cap C \neq \emptyset$). By inductive hypothesis, since $\mathcal{C}^{k+1} \subseteq \mathcal{C}^k$ and $X^{k+1} \subseteq X^k$, π^k satisfies the conservation law between C^1, C^2, C_1^2 , and C_2^1 . Thus, we conclude that $\pi_{C_1^2}^{k+1} + \pi_{C_2^1}^{k+1} = \pi_{C_1^2}^k + \pi_{C_2^1}^k - 2w^k = \pi_{C_1^2}^k + \pi_{C_2^1}^k$.

iv. This is a consequence of (i)–(iii); the proof is analogous to the one derived for $k = 1$.

Therefore, we conclude by induction that (i)–(iv) hold for every $k \in \llbracket 1, n^* + 1 \rrbracket$. \square

The proof of Proposition 2 highlights the importance of our construction of \mathcal{C}^{k+1} for $k \in \llbracket 1, n^* \rrbracket$ as given in (A9). This step of the algorithm ensures that \mathcal{C}^{k+1} preserves the decomposition of maximal chains intersecting in X^k . It also ensures that each maximal chain in \mathcal{C}^{k+1} intersects S^k . A direct consequence is that π^{k+1} satisfies the conservation law on the maximal chains of \mathcal{C}^{k+1} that intersect in X^k . This implies that \mathcal{C}^{k+1} preserves the decomposition of maximal chains intersecting in X^{k+1} , and P^{k+1} is a poset (Lemma 2). The issue, however, is that some maximal chains in \mathcal{C}^k may be removed when constructing \mathcal{C}^{k+1} , and we must ensure that the corresponding Constraints (6) will still be satisfied by the output of the algorithm. This is the focus of the next part.

3.2. Part 2: Feasibility of Algorithm 1’s Output

The second main part of the proof of Theorem 2 is to show that the algorithm terminates and outputs a feasible solution of (Q). Showing that the algorithm terminates is based on the fact that there is a finite number of elements and maximal chains. From (A10), we deduce that Constraints (5) are automatically satisfied at termination, since an element $x \in X$ is removed whenever the remaining value ρ_x^k is zero. Similarly, from Proposition 2, we obtain that Constraints (6) are satisfied for all maximal chains in \mathcal{C}^{n^*+1} , that is, the maximal chains that are not removed by the algorithm. For the remaining maximal chains $C \in \mathcal{C} \setminus \mathcal{C}^{n^*+1}$, we create a finite sequence of “dominating” maximal chains and show that Constraint (6) being satisfied for the last maximal chain of the sequence implies that it is also satisfied for the initial maximal chain C . To carry out this argument, we essentially need the following lemma:

Lemma 3. Consider $C^{(1)} \in \mathcal{C}$, and suppose that $C^{(1)} \in \mathcal{C}^{k_1} \setminus \mathcal{C}^{k_1+1}$ and $C^{(1)} \cap X^{k_1} \neq \emptyset$ for some $k_1 \in \llbracket 1, n^* \rrbracket$. Then, there exists $C^{(2)} \in \mathcal{C}^{k_1+1}$ such that $\delta_{C^{(1)}}^{k_1} \geq \delta_{C^{(2)}}^{k_1}$ and $C^{(2)} \cap X^{k_1} \supseteq C^{(1)} \cap X^{k_1}$.

Proof of Lemma 3. Consider $C^{(1)} \in \mathcal{C}$, and suppose that there exists $k_1 \in \llbracket 1, n^* \rrbracket$ such that $C^{(1)} \in \mathcal{C}^{k_1} \setminus \mathcal{C}^{k_1+1}$ and $C^{(1)} \cap X^{k_1} \neq \emptyset$. This case arises when the minimal element of $C^{(1)} \cap X^{k_1}$ in P is not a minimal element of P^{k_1} . Then, there is a chain in P^{k_1} whose maximal element is the minimal element of $C^{(1)} \cap X^{k_1}$ in P , and whose minimal element is a minimal element of P^{k_1} . By definition of P^{k_1} , this chain is contained in a maximal chain in $\overline{\mathcal{C}}^{k_1}$ (Lemma 2). We then exploit (i)–(iii) in Proposition 2 to show that there exists a maximal chain in \mathcal{C}^{k_1+1} satisfying the desired properties.

Formally, let x^* denote the minimal element of $C^{(1)} \cap X^{k_1}$ in P . Since $C^{(1)} \notin \mathcal{C}^{k_1+1}$, then $x^* \notin S^{k_1}$. Let $C' \subseteq X^{k_1}$ denote a maximal chain of P^{k_1} that contains x^* . From Lemma 1, the minimal element of C' in P^{k_1} , which we denote y_1 , is a minimal element of P^{k_1} . Therefore, $y_1 \in S^k$ and $y_1 \neq x^*$. Thus, $|C'| \geq 2$, and there exists a maximal chain $C^2 \in \overline{\mathcal{C}}^{k_1}$ such that $C' = C^2 \cap X^{k_1}$ (Lemma 2). Since $C^{(1)} \cap C^2 \cap X^{k_1-1} \supseteq \{x^*\} \neq \emptyset$, let us consider the other two maximal chains $C_1^2, C_2^2 \in \mathcal{C}$ such that $C_1^2 \cup C_2^2 = C^{(1)} \cup C^2$. Since $C^{(1)}$ and C^2 are in \mathcal{C}^{k_1} , then from Proposition 2, C_1^2 and C_2^2 are in \mathcal{C}^{k_1} as well. Let us rewrite $C^{(1)} = \{x_{-m}, \dots, x_0 = x^*, \dots, x_n\}$, $C^2 = \{y_{-q}, \dots, y_0, y_1, \dots, y_p = x^*, \dots, y_{p+r}\}$, $C_1^2 = \{x_{-m}, \dots, x_{-1}, y_p, \dots, y_{p+r}\}$, and $C_2^2 = \{y_{-q}, \dots, y_p, x_1, \dots, x_n\}$; they are illustrated in Figure 4.

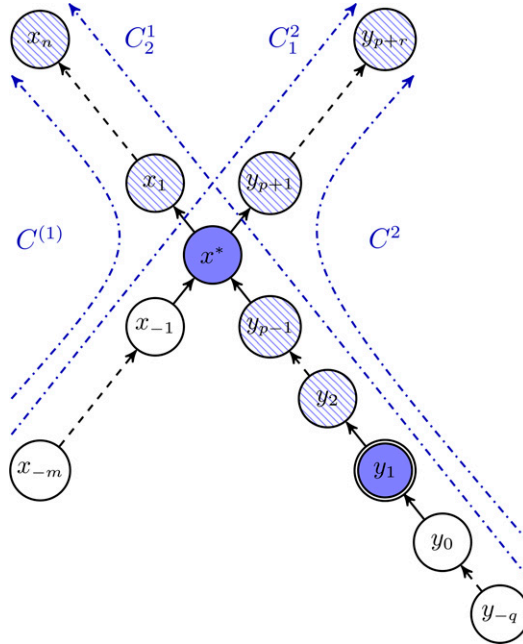
Since x^* is the minimal element of $C^{(1)} \cap X^{k_1}$ in P , then for all $i \in \llbracket -m, -1 \rrbracket$, $\rho_{x_i}^{k_1} = 0$. Because $C^2 \in \overline{\mathcal{C}}^{k_1}$ and $C_1^2 \in \mathcal{C}^{k_1}$, and from the conservation law between $C^{(1)}, C^2, C_1^2$, and C_2^2 , we obtain

$$\pi_{C_2^2}^{k_1} - \pi_{C^{(1)}}^{k_1} \stackrel{(12)}{=} \pi_{C^2}^{k_1} - \pi_{C_1^2}^{k_1} \stackrel{(10)}{=} \sum_{j=-q}^{p+r} \rho_{y_j}^{k_1} - \underbrace{\delta_{C^2}^{k_1}}_{=0} - \sum_{i=-m}^{-1} \underbrace{\rho_{x_i}^{k_1}}_{=0} - \sum_{j=p}^{p+r} \rho_{y_j}^{k_1} + \underbrace{\delta_{C_1^2}^{k_1}}_{\geq 0} \stackrel{(11)}{\geq} \sum_{j=-q}^{p-1} \rho_{y_j}^{k_1}. \quad (13)$$

This implies that

$$\delta_{C^{(1)}}^{k_1} \stackrel{(10)}{=} \sum_{i=0}^n \rho_{x_i}^{k_1} - \pi_{C^{(1)}}^{k_1} + \sum_{j=-q}^{p-1} \rho_{y_j}^{k_1} - \sum_{j=-q}^{p-1} \rho_{y_j}^{k_1} \stackrel{(10)}{=} \delta_{C_2^2}^{k_1} + \pi_{C_2^2}^{k_1} - \pi_{C^{(1)}}^{k_1} - \sum_{j=-q}^{p-1} \rho_{y_j}^{k_1} \stackrel{(13)}{\geq} \delta_{C_2^2}^{k_1}.$$

Figure 4. (Color online) Illustration of $C^{(1)}$, C^2 , C_1^2 , and C_2^1 . The filled nodes are the elements in X^{k_1} , the hatched nodes are the elements that may or may not be in X^{k_1} , and the white nodes are the elements that are not in X^{k_1} . The “double” node y_1 is in S^{k_1} .



Since y_1 is the minimal element of $C^2 \cap X^{k_1}$ in P^{k_1} , it is also the minimal element of $C^2 \cap X^{k_1}$ in P . Therefore, y_1 is the minimal element of $C_2^1 \cap X^{k_1}$ in P . Since $y_1 \in S^{k_1}$, then $C_2^1 \in \mathcal{C}^{k_1+1}$.

Finally, since for all $i \in \llbracket -m, -1 \rrbracket$, $x_i \notin X^{k_1}$, then $C_2^1 \cap X^{k_1} \supseteq \{x^*, x_1, \dots, x_n\} \cap X^{k_1} = C^{(1)} \cap X^{k_1}$, as illustrated in Figure 4. In conclusion, given $C^{(1)} \in \mathcal{C}^{k_1} \setminus \mathcal{C}^{k_1+1}$ such that $C^{(1)} \cap X^{k_1} \neq \emptyset$, there exists $C^{(2)} := C_2^1 \in \mathcal{C}^{k_1+1}$ such that $\delta_{C^{(1)}}^{k_1} \geq \delta_{C^{(2)}}^{k_1}$ and $C^{(2)} \cap X^{k_1} \supseteq C^{(1)} \cap X^{k_1}$. \square

As shown in the next proposition, one of Lemma 3’s implications is that if a maximal chain $C^{(1)}$ is removed after the k_1 -th iteration of the algorithm, then there exists another maximal chain $C^{(2)}$ that dominates $C^{(1)}$ in that if the output of the algorithm satisfies Constraint (6) for $C^{(2)}$, then it also satisfies that constraint for $C^{(1)}$. Additionally, it is guaranteed that $C^{(2)}$ is not removed before the $k_1 + 1$ -th iteration of the algorithm. We now show the feasibility of Algorithm 1’s output:

Proposition 3. Algorithm 1 terminates and outputs a feasible solution of (Q) .

Proof of Proposition 3. We recall that the algorithm terminates after iteration n^* if $X^{n^*+1} = \emptyset$. First, note that $X^1 \subseteq X$ and for all $k \in \llbracket 1, n^* \rrbracket$, $X^{k+1} \stackrel{(A10)}{\subseteq} X^k$. Additionally, $\hat{C}^1 \subseteq C$, and from (A8), $\hat{C}^{k+1} \subseteq \hat{C}^k$ for every $k \in \llbracket 1, n^* \rrbracket$. Now, consider $k \in \llbracket 1, n^* \rrbracket$, and the weight w^k chosen by the algorithm at iteration k . From (A7), there exists $x \in X^k$ such that $w^k = \rho_x^k$, or there exists $C \in \hat{C}^k$ such that $w^k = \frac{\delta_C^k}{|S^k \cap C| - 1}$. In the first case, $x \notin X^{k+1}$, so $X^{k+1} \subsetneq X^k$. In the second case, either $C \notin \mathcal{C}^{k+1}$ or $C \in \mathcal{C}^{k+1}$ with $\delta_C^{k+1} = 0$; this implies that $C \notin \hat{C}^{k+1}$ and $\hat{C}^{k+1} \subsetneq \hat{C}^k$.

Thus, for every $k \in \llbracket 1, n^* \rrbracket$, $|X^{k+1} \times \hat{C}^{k+1}| < |X^k \times \hat{C}^k|$. Since $|X^1 \times \hat{C}^1| \in \mathbb{N}$, if n^* were equal to $+\infty$, we would obtain an infinite decreasing sequence of natural integers. Therefore, we conclude that $n^* \in \mathbb{N}$, that is, the algorithm terminates. At termination, $X^{n^*+1} = \emptyset$.

Next, we show that the output $\sigma \in \mathbb{R}_{\geq 0}^P$ of the algorithm is a feasible solution of (Q) . First, the equality Constraints (5) are trivially satisfied:

$$\forall x \in X, \rho_x \stackrel{(A1)}{=} \rho_x^1 \stackrel{(A8)}{=} \underbrace{\rho_x^{n^*+1}}_{=0} + \sum_{k=1}^{n^*} w^k \mathbb{1}_{\{x \in S^k\}} \stackrel{(A7)}{=} \sum_{k=1}^{n^*} \sigma_{S^k} \mathbb{1}_{\{x \in S^k\}} = \sum_{\{S \in \mathcal{P} \mid x \in S\}} \sigma_S.$$

Regarding Constraints (6), we first show the following equality:

$$\forall C \in \mathcal{C}, \delta_C^{n^*+1} \stackrel{(A8)}{=} \delta_C^1 - \sum_{k=1}^{n^*} w^k (|S^k \cap C| - 1) \mathbb{1}_{\{|S^k \cap C| \geq 2\}} \stackrel{(A1), (A7)}{=} \delta_C - \sum_{\{S \in \mathcal{P} \mid |S \cap C| \geq 2\}} \sigma_S (|S \cap C| - 1).$$

Therefore, Constraints (6) are satisfied if and only if for every $C \in \mathcal{C}$, $\delta_C^{n^*+1} \geq 0$.

From Proposition 2, we know that for all $C \in \mathcal{C}^{n^*+1}$, $\delta_C^{n^*+1} \geq 0$. Now, consider $C^{(1)} \in \mathcal{C}$, and suppose that there exists $k_1 \in \llbracket 1, n^* \rrbracket$ such that $C^{(1)} \in \mathcal{C}^{k_1} \setminus \mathcal{C}^{k_1+1}$. If $C^{(1)} \cap X^{k_1} = \emptyset$, then for every $l \in \llbracket k_1, n^* \rrbracket$, $|S^l \cap C^{(1)}| = 0$ because $S^l \stackrel{(A6)}{\subseteq} X^l$ and $X^l \stackrel{(A10)}{\subseteq} X^{k_1}$. Therefore, because $C^{(1)} \in \mathcal{C}^{k_1}$, we have $\delta_{C^{(1)}}^{n^*+1} \stackrel{(A8)}{=} \delta_{C^{(1)}}^{k_1} - \sum_{l=k_1}^{n^*} w^l (|S^l \cap C^{(1)}| - 1) \mathbb{1}_{\{|S^l \cap C^{(1)}| \geq 2\}} = \delta_{C^{(1)}}^{k_1} \stackrel{(11)}{\geq} 0$.

If $C^{(1)} \cap X^{k_1} \neq \emptyset$, then there exists $C^{(2)} \in \mathcal{C}^{k_1+1}$ such that $\delta_{C^{(1)}}^{k_1} \geq \delta_{C^{(2)}}^{k_1}$ and $C^{(2)} \cap X^{k_1} \supseteq C^{(1)} \cap X^{k_1}$ (Lemma 3). For any $i \in \llbracket k_1, n^* \rrbracket$, $S^i \cap C^{(2)} \supseteq S^i \cap C^{(1)}$ because $S^i \stackrel{(A6), (A10)}{\subseteq} X^{k_1}$. Then, we obtain

$$\begin{aligned} \forall l \in \llbracket k_1, n^* + 1 \rrbracket, \delta_{C^{(1)}}^l &\stackrel{(A8)}{=} \delta_{C^{(1)}}^{k_1} - \sum_{i=k_1}^{l-1} w^i (|S^i \cap C^{(1)}| - 1) \mathbb{1}_{\{|S^i \cap C^{(1)}| \geq 2\}} \\ &\geq \delta_{C^{(2)}}^{k_1} - \sum_{i=k_1}^{l-1} w^i (|S^i \cap C^{(2)}| - 1) \mathbb{1}_{\{|S^i \cap C^{(2)}| \geq 2\}} \stackrel{(A8)}{=} \delta_{C^{(2)}}^l. \end{aligned} \quad (14)$$

In particular, $\delta_{C^{(1)}}^{n^*+1} \geq \delta_{C^{(2)}}^{n^*+1}$.

By induction, we construct a sequence of maximal chains $(C^{(s)})$, a sequence of increasing integers (k_s) , and a termination point $s^* \in \mathbb{N}$, such that for all $s \in \llbracket 1, s^* - 1 \rrbracket$, $C^{(s)} \in \mathcal{C}^{k_s} \setminus \mathcal{C}^{k_s+1}$, $\delta_{C^{(s)}}^{n^*+1} \geq \delta_{C^{(s+1)}}^{n^*+1}$, and $\delta_{C^{(s^*)}}^{n^*+1} \geq 0$. Note that s^* exists since $k_s \leq n^* + 1$. This implies that $\delta_{C^{(1)}}^{n^*+1} \geq \dots \geq \delta_{C^{(s^*)}}^{n^*+1} \geq 0$.

Thus, for every $C \in \mathcal{C}$, $\delta_C^{n^*+1} \geq 0$, and Constraints (6) are satisfied by the output σ of the algorithm. In conclusion, the algorithm outputs a feasible solution of (Q). \square

The output of Algorithm 1, by design, satisfies Constraints (5), and also Constraints (6) for the maximal chains in \mathcal{C}^{n^*+1} . Recall that the remaining maximal chains were removed after an iteration k in order to maintain the conservation law on the resulting set \mathcal{C}^{k+1} . This conservation law played an essential role in proving Proposition 3, that is, in showing that Constraints (6) are also satisfied for the maximal chains that are not in \mathcal{C}^{n^*+1} (see the proof of Lemma 3).

3.3. Part 3: Optimality of Algorithm 1

The final part of the proof of Theorem 2 consists in showing that the total weight used by the algorithm is exactly $\max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$. This is done by considering the following quantity: for every $k \in \llbracket 1, n^* + 1 \rrbracket$, $W^k := \max\{\max\{\rho_x^k, x \in X\}, \max\{\pi_C^k, C \in \mathcal{C}\}\}$. First, we show that for every $k \in \llbracket 1, n^* \rrbracket$, $W^{k+1} = W^k - w^k$. Then, we show that $W^{n^*+1} = 0$. Using a telescoping series, we obtain the desired result. Lemma 3 is also used to conclude that $\max\{\pi_C^k, C \in \mathcal{C}\}$ is attained by a maximal chain $C \in \mathcal{C}^{k+1}$.

Proposition 4. *The total weight used by Algorithm 1 when it terminates is $\max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$.*

Proof of Proposition 4. For all $k \in \llbracket 1, n^* + 1 \rrbracket$, let $W^k := \max\{\max\{\rho_x^k, x \in X\}, \max\{\pi_C^k, C \in \mathcal{C}\}\}$. First, we show that for every $k \in \llbracket 1, n^* \rrbracket$, $W^{k+1} = W^k - w^k$. Consider $k \in \llbracket 1, n^* \rrbracket$, and let $C \in \mathcal{C} \setminus \mathcal{C}^{k+1}$. Then, there exists $k_1 \leq k$ such that $C \in \mathcal{C}^{k_1} \setminus \mathcal{C}^{k_1+1}$. If $C \cap X^{k_1} = \emptyset$, then $\pi_C^{k+1} \stackrel{(9)}{\leq} \pi_C^k \stackrel{(9)}{\leq} \pi_C^{k_1} \stackrel{(10)}{=} -\delta_C^{k_1} \stackrel{(11)}{\leq} 0$. If $C \cap X^{k_1} \neq \emptyset$, then Lemma 3 implies that there exists $C^{(2)} \in \mathcal{C}^{k_1+1}$ such that for all $l \in \llbracket k_1, n^* + 1 \rrbracket$, $\delta_C^l \geq \delta_{C^{(2)}}^l$, and $C^{(2)} \cap X^l \supseteq C \cap X^l$. Consequently, we obtain

$$\forall l \in \llbracket k_1, n^* + 1 \rrbracket, \pi_C^{(10)} \sum_{x \in C \cap X^l} \rho_x^l - \delta_C^l + \pi_{C^{(2)}}^l + \delta_{C^{(2)}}^l - \sum_{x \in C \cap X^l} \rho_x^l - \sum_{x \in (C^{(2)} \cap X^l) \setminus (C \cap X^l)} \rho_x^l \stackrel{(14)}{\leq} \pi_{C^{(2)}}^l.$$

In particular, $\pi_C^k \leq \pi_{C^{(2)}}^k$ and $\pi_C^{k+1} \leq \pi_{C^{(2)}}^{k+1}$. As in Proposition 3, we construct a sequence of maximal chains $(C^{(s)})$, a sequence of increasing integers (k_s) , and a termination point $s' \in \mathbb{N}$, such that $C^{(1)} = C$ and for all $s \in \llbracket 1, s' - 1 \rrbracket$, $C^{(s)} \in \mathcal{C}^{k_s} \setminus \mathcal{C}^{k_s+1}$, $\pi_{C^{(s)}}^k \leq \pi_{C^{(s+1)}}^k$, and $\pi_{C^{(s)}}^{k+1} \leq \pi_{C^{(s+1)}}^{k+1}$. At termination, $C^{(s')} \in \mathcal{C}^{k_{s'}}$, and either $k_{s'} = k + 1$ or $k_{s'} < k + 1$ and $C^{(s')} \cap X^{k_{s'}} = \emptyset$. If $k_{s'} = k + 1$, then $\pi_C^k \leq \pi_{C^{(s')}}^k$ and $\pi_C^{k+1} \leq \pi_{C^{(s')}}^{k+1}$, with $C^{(s')} \in \mathcal{C}^{k+1}$. If $k_{s'} < k + 1$ and

$C^{(s')} \cap X^{k_s'} = \emptyset$, then $\pi_C^{k+1} \stackrel{(9)}{\leq} \pi_C^k \leq \pi_{C^{(s')}}^k \stackrel{(9)}{\leq} \pi_{C^{(s')}}^{k_s'} \stackrel{(10)}{=} -\delta_{C^{(s')}}^{k_s'} \stackrel{(11)}{\leq} 0 \leq \rho_x^{k+1} \leq \rho_x^k$ for all $x \in X$. Thus, $W^k = \max\{\max\{\rho_x^k, x \in X\}, \max\{\pi_C^k, C \in \mathcal{C}^{k+1}\}\}$, and $W^{k+1} = \max\{\max\{\rho_x^{k+1}, x \in X\}, \max\{\pi_C^{k+1}, C \in \mathcal{C}^{k+1}\}\}$.

Since $X^k \neq \emptyset$, $\rho_x^k \geq \rho_x^{k+1} \geq 0$ for every $x \in X^k$, and $\rho_x^k = \rho_x^{k+1} = 0$ for every $x \in X \setminus X^k$, then $\max\{\rho_x^k, x \in X\} = \max\{\rho_x^k, x \in X^k\}$, and $\max\{\rho_x^{k+1}, x \in X\} = \max\{\rho_x^{k+1}, x \in X^k\}$.

Next, let $x \in X^k \setminus S^k$. Then, there exists $y \in S^k$ such that $y \preceq_{\bar{C}^k} x$. By definition, there exists $C \in \bar{C}^k$ such that $y, x \in C$. In fact, y is the minimal element of $C \cap X^k$ in P^k , and $C \in \mathcal{C}^{k+1}$. Since $C \in \bar{C}^k$, then $\pi_C^k \stackrel{(10)}{=} \sum_{x' \in C \cap P_{x'}} \rho_{x'}^k \geq \rho_x^k + \rho_y^k \geq \rho_x^k$. Furthermore, since $y \in S^k$, then $w^k \stackrel{(A7)}{\leq} \rho_y^k$. Thus, $\rho_x^{k+1} = \rho_x^k \leq \pi_C^k - \rho_y^k \leq \pi_C^k - w^k \stackrel{(9)}{=} \pi_C^{k+1}$, from which we conclude that $W^k = \max\{\max\{\rho_x^k, x \in S^k\}, \max\{\pi_C^k, C \in \mathcal{C}^{k+1}\}\}$ and $W^{k+1} = \max\{\max\{\rho_x^{k+1}, x \in S^k\}, \max\{\pi_C^{k+1}, C \in \mathcal{C}^{k+1}\}\}$.

Finally, we note that for all $C \in \mathcal{C}^{k+1}$, $\pi_C^{k+1} \stackrel{(9)}{=} \pi_C^k - w^k$ because $S^k \cap C \neq \emptyset$, and for all $x \in S^k$, $\rho_x^{k+1} \stackrel{(A8)}{=} \rho_x^k - w^k$. Putting everything together, we conclude

$$\begin{aligned} W^{k+1} &= \max\{\max\{\rho_x^{k+1}, x \in S^k\}, \max\{\pi_C^{k+1}, C \in \mathcal{C}^{k+1}\}\} \\ &= \max\{\max\{\rho_x^k, x \in S^k\}, \max\{\pi_C^k, C \in \mathcal{C}^{k+1}\}\} - w^k = W^k - w^k. \end{aligned}$$

Next, we show that $W^{n^*+1} = 0$. First, $\rho_x^{n^*+1} = 0$ for all $x \in X$. Secondly, $\pi_C^{n^*+1} \stackrel{(10)}{=} -\delta_C^{n^*+1} \stackrel{(11)}{\leq} 0$ for all $C \in \mathcal{C}^{n^*+1}$. Thirdly, $S^{n^*} \neq \emptyset$ because $X^{n^*} \neq \emptyset$. This implies that $W^{n^*+1} = \max\{\max\{\rho_x^{n^*+1}, x \in S^{n^*}\}, \max\{\pi_C^{n^*+1}, C \in \mathcal{C}^{n^*+1}\}\} = 0$. Finally, using a telescoping series, we obtain

$$\sum_{S \in \mathcal{P}} \sigma_S \stackrel{(A7)}{=} \sum_{k=1}^{n^*} (W^k - W^{k+1}) = W^1 - \underbrace{W^{n^*+1}}_{=0} \stackrel{(A1),(9)}{=} \max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}. \quad \square$$

In conclusion, Propositions 2, 3, and 4 enable us to show that Algorithm 1 outputs a feasible solution of (Q) with objective value equal to $\max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$. Therefore $z_{(Q)}^* \leq \max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$. Since we already established the reverse inequality at the end of Section 2.3, we conclude that $z_{(Q)}^* = \max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$, thus proving Theorem 2.

Furthermore, since $\rho_x \leq 1$ for every $x \in X$, and $\pi_C \leq 1$ for every $C \in \mathcal{C}$, then $z_{(Q)}^* \leq 1$. This implies that (D) is feasible: Given the output σ of Algorithm 1, the vector $\hat{\sigma} \in \mathbb{R}_{\geq 0}^{\mathcal{P}}$ obtained from σ by additionally assigning $1 - z_{(Q)}^*$ to \emptyset is a feasible solution of Problem (D) and proves Theorem 1.

In fact, (Q) is a generalization of the minimum-weighted fractional coloring problem on comparability graphs (Hoàng [20]). The comparability graph of the poset $P = (X, \preceq)$ is an undirected graph whose set of nodes is X and whose edges are given by the pairs of comparable elements in P . In the special case where for all $C \in \mathcal{C}$, $\sum_{x \in C} \rho_x = \pi_C$ (i.e., Inequalities (2) are tight), (Q) is equivalent to the minimum-weighted fractional coloring problem on the comparability graph of P . Algorithm 1 can then be refined into Hoàng's $O(|X|^2)$ -time algorithm.

Given E_P the edge set of the cover graph of P (as defined in Section 2.1), the number of iterations of Algorithm 1 is upper bounded by $|X| + |E_P|$. However, Algorithm 1 requires at each iteration k the storage of the possibly exponentially many maximal chains C in \mathcal{C}^k , along with their corresponding values δ_C^k . Next, we develop an efficient implementation of Algorithm 1 when π is an affine function of the elements constituting each maximal chain of P .

4. Affine Case: Polynomial Algorithm

Consider Problem (D) for a given poset $P = (X, \preceq)$ and vectors $\rho \in [0,1]^X$ and $\pi \in (-\infty,1]^{\mathcal{C}}$ satisfying (2). In addition, we assume that the value of each maximal chain $C \in \mathcal{C}$ in P is given by $\pi_C = \alpha - \sum_{x \in C} \beta_x$, with $\alpha \in \mathbb{R}$ and $\beta_x \in \mathbb{R}$ for every $x \in X$. We observe that π satisfies the conservation law (3), and (D) is feasible. In this section, we refine Algorithm 1 for this special case and show that an optimal solution of (Q) can be computed in polynomial time.

Our polynomial algorithm performs (A5) and (A7) without enumerating all the maximal chains of P . Instead, it runs subroutines based on the shortest path algorithm in the directed cover graph of P to construct the subposet P^k and compute the maximum weight w^k that can be assigned at each iteration $k \in \llbracket 1, n^* \rrbracket$. Let us discuss the execution of the first iteration of the algorithm.

Firstly, we augment the poset P by adding an artificial source element s and destination element t that satisfy $s \preceq x \preceq t$, for every $x \in X$; let P^* denote the augmented poset. Then, the algorithm stores the directed cover graph $H_{P^*} = (X \cup \{s, t\}, E_{P^*})$ of P^* . An $s-t$ path of size n is a sequence of edges $\{e_1 = (s_1, t_1), \dots, e_n = (s_n, t_n)\}$ such that $s_1 = s$, $t_n = t$, and for all $i \in \llbracket 1, n-1 \rrbracket$, $t_i = s_{i+1}$. Note that the set of maximal chains \mathcal{C} of P is equivalent to the set of $s-t$ paths of H_{P^*} : The set of nodes in X visited by an $s-t$ path is a maximal chain $C \in \mathcal{C}$ and vice versa.

Next, the algorithm sets the length $\rho_y + \beta_y$ to every edge $(x, y) \in E_{P^*}$ with $y \neq t$ and sets the length $-\alpha$ to every edge $(x, t) \in E_{P^*}$ with $x \in X$. Thus, the length of every $s - t$ path in H_{P^*} , whose corresponding maximal chain is $C \in \mathcal{C}$, is $\sum_{x \in C} (\rho_x + \beta_x) - \alpha \stackrel{(4)}{=} \delta_C$. We then compute the shortest distances between all pairs of nodes in H_{P^*} : We first topologically sort H_{P^*} , and then run the classical shortest path algorithm in directed acyclic graphs starting from each node of H_{P^*} . We store the shortest distances in a matrix $M = (m_{xy})_{(x,y) \in (X \cup \{s,t\})^2}$. By definition of $P^1 = (X^1, \preceq_{\bar{C}^1})$ in Algorithm 1, and because $\delta_C \stackrel{(2)}{\geq} 0$ for every $C \in \mathcal{C}$, we obtain that for every $(x, y) \in (X^1)^2$ with $x \neq y$, $x \preceq_{\bar{C}^1} y$ if and only if the length $m_{sx} + m_{xy} + m_{yt}$ of a shortest $s - t$ path in H_{P^*} that goes through x and y is zero. Thus, this shortest path subroutine replaces (A5) and constructs P^1 in polynomial time. The algorithm then selects its set of minimal elements S^1 .

Now, we compute the weight w^1 to assign to S^1 without enumerating all maximal chains $C \in \hat{\mathcal{C}}^1$ such that $|S^1 \cap C| \geq 2$: Our algorithm constructs the subposet $\hat{P}^1 := (S^1 \cup \{s, t\}, \preceq|_{S^1 \cup \{s,t\}})$ of P^* and stores its directed cover graph $H_{\hat{P}^1} = (S^1 \cup \{s, t\}, E_{\hat{P}^1})$. The length of each edge $(x, y) \in E_{\hat{P}^1}$ is set to the shortest distance m_{xy} from x to y in the graph H_{P^*} . Then, we extend the shortest path algorithm in directed acyclic graphs to obtain, for each $q \in \llbracket 1, |S^1| \rrbracket$, the distance $\hat{\ell}^q$ of a shortest path from s to t that traverses q elements of S^1 . The maximum weight to assign to S^1 can be efficiently computed as $w^1 = \min\left\{\min\{\rho_x, x \in S^1\}, \min\left\{\frac{\hat{\ell}^q}{q-1}, q \in \llbracket 2, |S^1| \rrbracket\right\}\right\}$, which replaces (A7).

Finally, the algorithm updates the vector ρ and the set of elements with positive ρ . In addition, w^1 must be subtracted from the scalar α to capture the update (9). This in turn will change the lengths of the edges in H_{P^*} for the next iteration. The key challenges for the analysis of the subsequent iterations k are to account for the fact that some maximal chains are removed by Algorithm 1, and that the length of an $s - t$ path in H_{P^*} , whose corresponding maximal chain is $C \in \mathcal{C}$, is not necessarily δ_C^k . We now formally present Algorithm 2.

Algorithm 2 (Optimal Solution of (Q) in Affine Case)

Input: Finite nonempty poset $P = (X, \preceq)$, scalar $\alpha \in \mathbb{R}$, and vectors $\rho \in \mathbb{R}_{\geq 0}^X$, $\beta \in \mathbb{R}^X$.

Output: Vector $\tilde{\sigma} \in \mathbb{R}_{\geq 0}^P$.

B1: Augment the poset into $P^* = (X \cup \{s, t\}, \preceq)$, where $s \preceq x \preceq t$, $\forall x \in X$

B2: Construct the directed cover graph $H_{P^*} = (X \cup \{s, t\}, E_{P^*})$

B3: $\tilde{\rho}_x^1 \leftarrow \rho_x$, $\forall x \in X$, $\tilde{\rho}_t^1 \leftarrow 0$, $\beta_x^1 \leftarrow \beta_x$, $\forall x \in X$, $\beta_t^1 \leftarrow -\alpha$, $\tilde{X}^1 \leftarrow \{x \in X \mid \tilde{\rho}_x^1 > 0\}$

B4: $k \leftarrow 1$

B5: **while** $\tilde{X}^k \neq \emptyset$ **do**

B6: Set the length of every edge $(x, y) \in E_{P^*}$ to $\tilde{\rho}_y^k + \beta_y^k$

B7: $M^k = (m_{xy}^k)_{(x,y) \in (X \cup \{s,t\})^2} \leftarrow$ all-pairs shortest distance matrix for the graph H_{P^*}

B8: Construct the poset $\tilde{P}^k = (\tilde{X}^k, \preceq_k)$: $\forall x, y \in \tilde{X}^k$ with $x \neq y$, $x \prec_k y \iff m_{sx}^k + m_{xy}^k + m_{yt}^k = 0$

B9: Select \tilde{S}^k the set of minimal elements of \tilde{P}^k

B10: Construct the subposet $\hat{P}^k = (\tilde{S}^k \cup \{s, t\}, \preceq|_{\tilde{S}^k \cup \{s,t\}})$ of P^*

B11: Construct the directed cover graph $H_{\hat{P}^k} = (\tilde{S}^k \cup \{s, t\}, E_{\hat{P}^k})$, and topologically sort it

B12: $\hat{\ell}_x^{k,q} \leftarrow +\infty$, $\forall x \in X \cup \{s, t\}$, $\forall q \in \llbracket -1, |\tilde{S}^k| \rrbracket$, $\hat{\ell}_s^{k,-1} \leftarrow 0$

B13: **for** $x \in \tilde{S}^k \cup \{s\}$ in topologically sorted order, and $y \in \tilde{S}^k \cup \{t\}$ such that $(x, y) \in E_{\hat{P}^k}$ **do**

B14: **for** $q \in \llbracket -1, |\tilde{S}^k| - 1 \rrbracket$ such that $\hat{\ell}_y^{k,q+1} > \hat{\ell}_x^{k,q} + m_{xy}^k$ **do**

B15: $\hat{\ell}_y^{k,q+1} \leftarrow \hat{\ell}_x^{k,q} + m_{xy}^k$

B16: **end for**

B17: **end for**

B18: $\tilde{w}^k \leftarrow \min\{\min\{\tilde{\rho}_x^k, x \in \tilde{S}^k\}, \min\{\frac{\hat{\ell}_t^{k,q}}{q-1}, q \in \llbracket 2, |\tilde{S}^k| \rrbracket\}\}$, $\tilde{\sigma}_{\tilde{S}^k} \leftarrow \tilde{w}^k$

B19: $\beta_x^{k+1} \leftarrow \beta_x^k$, $\forall x \in X$, $\beta_t^{k+1} \leftarrow \beta_t^k + \tilde{w}^k$, $\tilde{\rho}_x^{k+1} \leftarrow \tilde{\rho}_x^k - \tilde{w}^k \mathbb{1}_{\{x \in \tilde{S}^k\}}$, $\forall x \in X \cup \{t\}$

B20: $\tilde{X}^{k+1} \leftarrow \{x \in \tilde{X}^k \mid \tilde{\rho}_x^{k+1} > 0\}$

B21: $k \leftarrow k + 1$

B22: **end while**

For every maximal chain $C \in \mathcal{C}$, we define the following sequence induced by Algorithm 2, which represents the length of the corresponding $s - t$ path in H_{P^*} at the beginning of each iteration:

$$\forall k \in \llbracket 1, n^* + 1 \rrbracket, \quad \ell_C^k := \sum_{x \in C} (\tilde{\rho}_x^k + \beta_x^k) + \beta_t^k. \tag{15}$$

We now proceed with proving by induction that Algorithm 2 is a refinement of Algorithm 1:

Proposition 5. *Algorithm 1’s and Algorithm 2’s outputs are identical. In particular, for every $k \in \llbracket 1, n^* + 1 \rrbracket$, the following hold:*

- i. *The remaining values for each element are identical: for every $x \in X$, $\tilde{\rho}_x^k = \rho_x^k$, and $\tilde{X}^k = X^k$.*
- ii. *For every maximal chain $C \in \mathcal{C}$, the length of its corresponding $s - t$ path in H_{P^*} is at least δ_C^k . Furthermore, this inequality is tight for every maximal chain in \mathcal{C}^k :*

$$\forall C \in \mathcal{C}, \quad \ell_C^k \geq \delta_C^k, \tag{16}$$

$$\forall C \in \mathcal{C}^k, \quad \ell_C^k = \delta_C^k. \tag{17}$$

iii. $\tilde{P}^k = (\tilde{X}^k, \preceq_k)$ is a poset identical to $P^k = (X^k, \preceq_{\bar{C}^k})$, and $\tilde{S}^k = S^k$.

iv. *The weights assigned by both algorithms are identical: $\tilde{w}^k = w^k$.*

Proof of Proposition 5. We show (i)–(iv) by induction. First, consider $k = 1$. Since $\tilde{\rho} = \rho = \rho^1$, then $\tilde{X}^1 = X^1$. Furthermore, for all $C \in \mathcal{C}$, $\ell_C^1 \stackrel{(15)}{=} \sum_{x \in C} (\rho_x + \beta_x) - \alpha \stackrel{(4)}{=} \delta_C^1 \stackrel{(2)}{\geq} 0$. Therefore, for every $(x, y) \in (X^1)^2$ with $x \neq y$, $x \preceq_{\bar{C}^1} y$ if and only if there exists $C \in \bar{\mathcal{C}}^1$ such that $x, y \in C$, which in turn is equivalent to the length of a shortest path in H_{P^*} traversing x and y being zero. Thus, $\tilde{P}^1 = P^1$, and $\tilde{S}^1 = S^1$. Next, because $\ell^1 = \delta^1$, we obtain that

$$\begin{aligned} \tilde{w}^1 &\stackrel{(B18)}{=} \min \left\{ \min \{ \tilde{\rho}_x^1, x \in \tilde{S}^1 \}, \min \left\{ \frac{\hat{\ell}_t^{1,q}}{q-1}, q \in \llbracket 2, |\tilde{S}^1| \rrbracket \right\} \right\} \\ &= \min \left\{ \min \{ \rho_x^1, x \in S^1 \}, \min \left\{ \frac{\delta_C^1}{|S^1 \cap C| - 1}, C \in \mathcal{C} \mid |S^1 \cap C| \geq 2 \right\} \right\}. \end{aligned}$$

Note that $|S^1 \cap C| \leq 1$ for every maximal chain $C \in \bar{\mathcal{C}}^1$, by definition of P^1 . Since $\mathcal{C}^1 = \mathcal{C}$, we deduce that $\{C \in \mathcal{C} \mid |S^1 \cap C| \geq 2\} = \{C \in \hat{\mathcal{C}}^1 \mid |S^1 \cap C| \geq 2\}$. Therefore,

$$\tilde{w}^1 = \min \left\{ \min \{ \rho_x^1, x \in S^1 \}, \min \left\{ \frac{\delta_C^1}{|S^1 \cap C| - 1}, C \in \hat{\mathcal{C}}^1 \mid |S^1 \cap C| \geq 2 \right\} \right\} \stackrel{(A7)}{=} w^1.$$

We now assume that (i)–(iv) hold for $k \in \llbracket 1, n^* \rrbracket$ and show that they also hold for $k + 1$:

i. Since $\tilde{\rho}^k = \rho^k$, $\tilde{S}^k = S^k$, and $\tilde{w}^k = w^k$, then for every $x \in X$, $\tilde{\rho}_x^{k+1} \stackrel{(B19)}{=} \tilde{\rho}_x^k - \tilde{w}^k \mathbb{1}_{\{x \in \tilde{S}^k\}} = \rho_x^k - w^k \mathbb{1}_{\{x \in S^k\}} \stackrel{(A8)}{=} \rho_x^{k+1}$. This also implies that $\tilde{X}^{k+1} = X^{k+1}$.

ii. For every maximal chain $C \in \mathcal{C}$, $\ell_C^k \geq \delta_C^k$ implies that

$$\ell_C^{k+1} \stackrel{(15), (B19)}{=} \ell_C^k - \tilde{w}^k |S^k \cap C| + \tilde{w}^k \geq \delta_C^k - w^k |S^k \cap C| + w^k \mathbb{1}_{\{S^k \cap C \neq \emptyset\}} \stackrel{(A8)}{=} \delta_C^{k+1}. \tag{18}$$

If $C \in \mathcal{C}^{k+1}$, then $S^k \cap C \neq \emptyset$. Since $\mathcal{C}^{k+1} \subseteq \mathcal{C}^k$, then $\ell_C^k = \delta_C^k$ by inductive hypothesis. Therefore, (18) is tight for $C \in \mathcal{C}^{k+1}$.

iii. Consider $(x, y) \in (X^{k+1})^2$ with $x \neq y$. If $x \preceq_{\bar{C}^{k+1}} y$, then there exists $C^* \in \bar{\mathcal{C}}^{k+1} \subseteq \mathcal{C}^{k+1}$ such that $x, y \in C^*$. Consequently, $\ell_{C^*}^{k+1} \stackrel{(17)}{=} \delta_{C^*}^{k+1} = 0 \leq \delta_{C^*}^{k+1} \stackrel{(16)}{\leq} \ell_{C^*}^{k+1}$ for every $C \in \mathcal{C}$. Therefore, the $s - t$ path corresponding to C^* is a shortest path in H_{P^*} that goes through x and y and has length zero. Thus, $x \preceq_{k+1} y$.

Now, assume that x and y are not comparable in P^{k+1} . Two cases arise:

Case 1. x and y are not comparable in P . Then, there is no $s - t$ path in H_{P^*} that goes through x and y , which implies that x and y are not comparable in \tilde{P}^{k+1} .

Case 2. $x \prec y$ in P . Then, $\delta_C^{k+1} > 0$ for all $C \in \mathcal{C}^{k+1}$ such that $x, y \in C$, by definition of P^{k+1} . Let $C' \in \mathcal{C}$ be the maximal chain corresponding to a shortest path in H_{P^*} that goes through x and y . If $C' \in \mathcal{C}^{k+1}$, then $\ell_{C'}^{k+1} \stackrel{(17)}{=} \delta_{C'}^{k+1} > 0$. If $C' \notin \mathcal{C}^{k+1}$, then by applying Lemma 3 as in Section 3, we obtain that there exists a maximal chain $C^{(2)} \in \mathcal{C}^{k+1}$ such that $\delta_{C^{(2)}}^{k+1} \geq \delta_{C^{(2)}}^{k+1}$ and $C^{(2)} \cap X^{k+1} \supseteq C' \cap X^{k+1}$. Consequently, $x, y \in C^{(2)}$ and $\ell_{C^{(2)}}^{k+1} \stackrel{(16)}{\geq} \delta_{C^{(2)}}^{k+1} \geq \delta_{C^{(2)}}^{k+1} > 0$. Thus, x and y are not comparable in \tilde{P}^{k+1} .

In conclusion, $\tilde{P}^{k+1} = P^{k+1}$ and $\tilde{S}^{k+1} = S^{k+1}$.

iv. First, we note that

$$\min \left\{ \frac{\ell_C^{k+1,q}}{q-1}, q \in \llbracket 2, |\tilde{S}^{k+1}| \rrbracket \right\} = \min \left\{ \frac{\ell_C^{k+1}}{|S^{k+1} \cap C| - 1}, C \in \mathcal{C} \mid |S^{k+1} \cap C| \geq 2 \right\}. \quad (19)$$

If the minimization problem in (19) is infeasible, that is, there is no maximal chain $C \in \mathcal{C}$ such that $|S^{k+1} \cap C| \geq 2$, then $\{C \in \hat{\mathcal{C}}^{k+1} \mid |S^{k+1} \cap C| \geq 2\} = \emptyset$. In this case, we obtain $\tilde{w}^{k+1} \stackrel{(B18)}{=} \min\{\tilde{\rho}_x^{k+1}, x \in \tilde{S}^k\} = \min\{\rho_x^{k+1}, x \in S^k\} \stackrel{(A7)}{=} w^{k+1}$.

Next, consider the case where (19) is feasible. We now show that the optimal value of (19) is achieved by a maximal chain in \mathcal{C}^{k+1} : Let $C^* \in \mathcal{C}$ be an optimal solution of (19) and assume that $C^* \notin \mathcal{C}^{k+1}$. Since $C^* \cap X^{k+1} \supseteq C^* \cap S^{k+1} \neq \emptyset$, then by applying Lemma 3, there exists a maximal chain $C^{(2)} \in \mathcal{C}^{k+1}$ such that $\delta_{C^*}^{k+1} \geq \delta_{C^{(2)}}^{k+1}$ and $C^{(2)} \cap S^{k+1} \supseteq C^* \cap S^{k+1}$. Therefore, we obtain

$$\frac{\ell_{C^*}^{k+1}}{|S^{k+1} \cap C^*| - 1} \stackrel{(16)}{\geq} \frac{\delta_{C^*}^{k+1}}{|S^{k+1} \cap C^*| - 1} \geq \frac{\delta_{C^{(2)}}^{k+1}}{|S^{k+1} \cap C^{(2)}| - 1} \stackrel{(17)}{=} \frac{\ell_{C^{(2)}}^{k+1}}{|S^{k+1} \cap C^{(2)}| - 1}.$$

Thus, $C^{(2)} \in \mathcal{C}^{k+1}$ is also an optimal solution of (19). Since $|C^{(2)} \cap S^{k+1}| \geq |C^* \cap S^{k+1}| \geq 2$, then by definition of P^{k+1} , $\delta_{C^{(2)}}^{k+1} > 0$ and $C^{(2)} \in \hat{\mathcal{C}}^{k+1}$. Then, we derive the following inequality:

$$\forall C \in \hat{\mathcal{C}}^{k+1} \mid |S^{k+1} \cap C| \geq 2, \frac{\delta_C^{k+1}}{|S^{k+1} \cap C| - 1} \stackrel{(17)}{=} \frac{\ell_C^{k+1}}{|S^{k+1} \cap C| - 1} \geq \frac{\ell_{C^{(2)}}^{k+1}}{|S^{k+1} \cap C^{(2)}| - 1} \stackrel{(17)}{=} \frac{\delta_{C^{(2)}}^{k+1}}{|S^{k+1} \cap C^{(2)}| - 1}.$$

Therefore, $C^{(2)} \in \arg \min \left\{ \frac{\delta_C^{k+1}}{|S^{k+1} \cap C| - 1}, C \in \hat{\mathcal{C}}^{k+1} \mid |S^{k+1} \cap C| \geq 2 \right\}$, and we obtain

$$\begin{aligned} \tilde{w}^{k+1} &\stackrel{(B18)}{=} \min \left\{ \min\{\tilde{\rho}_x^{k+1}, x \in \tilde{S}^{k+1}\}, \min \left\{ \frac{\ell_C^{k+1}}{|S^{k+1} \cap C| - 1}, C \in \mathcal{C} \mid |S^{k+1} \cap C| \geq 2 \right\} \right\} \\ &\stackrel{(17)}{=} \min \left\{ \min\{\rho_x^{k+1}, x \in S^{k+1}\}, \min \left\{ \frac{\delta_C^{k+1}}{|S^{k+1} \cap C| - 1}, C \in \hat{\mathcal{C}}^{k+1} \mid |S^{k+1} \cap C| \geq 2 \right\} \right\} \stackrel{(A7)}{=} w^{k+1}. \end{aligned}$$

We conclude by induction that (i)–(iv) hold for every $k \in \llbracket 1, n^* + 1 \rrbracket$. \square

In conclusion, Algorithm 2 computes an optimal solution of (\mathcal{Q}) when π is an affine function of the elements constituting each maximal chain. Importantly, Algorithm 2 is a polynomial algorithm: Its running time is governed by (B7) and (B13)–(B17), which both require $O(|X||E_P|)$ operations because H_{P^*} and $H_{\tilde{P}^k}$ are directed acyclic graphs (Ahuja et al. [2]). Since the algorithm terminates after $n^* \leq |X| + |E_P|$ iterations, Algorithm 2 runs in $O(|X||E_P|^2)$ time.

5. Applications to Network Interdiction

In this section, we introduce a strategic interdiction game involving a routing entity and an interdictor interacting on a flow network. We use Theorems 1 and 2 on the existence of probability distributions over posets to characterize the equilibria of this game. We also provide a solution approach for the equilibrium computation of this game, which involves solving a minimum-cost circulation problem and running Algorithm 2.

5.1. Game-Theoretic Model

Consider a flow network, modeled as a simple directed acyclic connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} (respectively, \mathcal{E}) represents the set of nodes (respectively, the set of edges) of the network. For each edge $(i, j) \in \mathcal{E}$, let $c_{ij} \in \mathbb{R}_{>0}$ denote its capacity. Assume that a single commodity can be sent through \mathcal{G} from a source node $s \in \mathcal{V}$ to a destination node $t \in \mathcal{V}$. Let Λ denote the set containing all $s - t$ paths of \mathcal{G} .

A flow, denoted by the vector $f \in \mathbb{R}_{\geq 0}^\Lambda$, enters the network from s and leaves from t . A flow f is said to be feasible if the flow through each edge does not exceed its capacity; that is, for all $(i, j) \in \mathcal{E}$, $f_{ij} := \sum_{\{\lambda \in \Lambda \mid (i, j) \in \lambda\}} f_\lambda \leq c_{ij}$. Let \mathcal{F} denote the set of feasible flows of \mathcal{G} . Given a feasible flow $f \in \mathcal{F}$, let $F(f) := \sum_{\lambda \in \Lambda} f_\lambda$ denote the amount of flow sent from the node s to the node t . Each edge $(i, j) \in \mathcal{E}$ is associated with a marginal transportation cost, denoted $b_{ij} \in \mathbb{R}_{>0}$. For each $s - t$ path $\lambda \in \Lambda$, $b_\lambda := \sum_{(i, j) \in \lambda} b_{ij}$ represents the cost of transporting one unit of flow through λ . Given a feasible flow $f \in \mathcal{F}$, $T(f) := \sum_{\lambda \in \Lambda} b_\lambda f_\lambda$ denotes the total transportation cost of f .

We define a two-player game $\Gamma := \langle \{1, 2\}, (\mathcal{F}, \mathcal{I}), (u_1, u_2) \rangle$ on the flow network \mathcal{G} . Player 1 (**P1**) is the routing entity that chooses to route a flow $f \in \mathcal{F}$ of goods through the network, and Player 2 (**P2**) is the interdictor who simultaneously chooses a subset of edges $I \in 2^{\mathcal{E}}$ to interdict. The action set for **P1** (respectively, **P2**) is \mathcal{F} (respectively, $\mathcal{I} := 2^{\mathcal{E}}$). For every edge $(i, j) \in \mathcal{E}$, $d_{ij} \in \mathbb{R}_{>0}$ denotes the cost of interdicting (i, j) . Thus, the cost of an interdiction $I \in \mathcal{I}$ is given by $C(I) := \sum_{(i,j) \in I} d_{ij}$. In this model, **P2** (respectively, **P1**) gains (respectively, loses) the flow that crosses the edges that are interdicted by **P2**. The model captures strategic routing situations when **P1** cannot observe **P2**'s actions before sending its flow and cannot reroute its flow after the interdiction. We do not consider partial edge interdictions for the sake of simplicity. The *effective flow* when a flow f is chosen by **P1** and an interdiction I is chosen by **P2** is f^I , where $f_\lambda^I = f_\lambda \mathbb{1}_{\{\lambda \cap I = \emptyset\}}$ for all $\lambda \in \Lambda$. We also suppose that the transportation cost incurred by **P1** is for the initial flow f and not for the effective flow f^I . This modeling choice reflects an ex ante monetary fee paid by **P1** to the network owner who provides **P1** the access to send a quantity of flow through the network.

The payoff of **P1** is the value of effective flow assessed by **P1** net the cost of transporting the initial flow: $u_1(f, I) = p_1 F(f^I) - T(f)$, where $p_1 \in \mathbb{R}_{>0}$ is the marginal value of effective flow for **P1**. Similarly, the payoff of **P2** is the value of interdicted flow assessed by **P2** net the cost of interdiction: $u_2(f, I) = p_2 (F(f) - F(f^I)) - C(I)$, where $p_2 \in \mathbb{R}_{>0}$ is the marginal value of interdicted flow for **P2**.

In playing the game Γ , **P1** can route goods in the network using a flow f realized from a chosen probability distribution on the set \mathcal{F} and **P2** can interdict subsets of edges according to a probability distribution on the set \mathcal{I} . Specifically, **P1** and **P2** respectively choose a mixed routing strategy $\sigma^1 \in \Delta(\mathcal{F})$ and a mixed interdiction strategy $\sigma^2 \in \Delta(\mathcal{I})$, where $\Delta(\mathcal{F}) = \{\sigma^1 \in \mathbb{R}_{\geq 0}^{\mathcal{F}} \mid \sum_{f \in \mathcal{F}} \sigma_f^1 = 1\}$, and $\Delta(\mathcal{I}) = \{\sigma^2 \in \mathbb{R}_{\geq 0}^{\mathcal{I}} \mid \sum_{I \in \mathcal{I}} \sigma_I^2 = 1\}$ denote the strategy sets. Here, σ_f^1 (respectively, σ_I^2) represents the probability assigned to the flow f (respectively, interdiction I) by **P1**'s routing strategy σ^1 (respectively, **P2**'s interdiction strategy σ^2). The players' strategies are independent randomizations. Given a strategy profile $\sigma = (\sigma^1, \sigma^2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{I})$, the expected payoffs are expressed as

$$U_1(\sigma^1, \sigma^2) = p_1 \mathbb{E}_\sigma[F(f^I)] - \mathbb{E}_\sigma[T(f)], \tag{20}$$

$$U_2(\sigma^1, \sigma^2) = p_2 (\mathbb{E}_\sigma[F(f)] - \mathbb{E}_\sigma[F(f^I)]) - \mathbb{E}_\sigma[C(I)]. \tag{21}$$

We will also use the notations $U_i(\sigma^1, I) = U_i(\sigma^1, \mathbb{1}_{\{I\}})$ and $U_i(f, \sigma^2) = U_i(\mathbb{1}_{\{f\}}, \sigma^2)$ for $i \in \{1, 2\}$. We focus on characterizing the mixed strategy Nash equilibria of the game $\langle \{1, 2\}, (\Delta(\mathcal{F}), \Delta(\mathcal{I})), (U_1, U_2) \rangle$. A strategy profile $(\sigma^1, \sigma^2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{I})$ is a mixed strategy *Nash Equilibrium* of game Γ if for all $\sigma^1 \in \Delta(\mathcal{F})$, $U_1(\sigma^1, \sigma^2) \geq U_1(\sigma^1, \sigma^2)$ and for all $\sigma^2 \in \Delta(\mathcal{I})$, $U_2(\sigma^1, \sigma^2) \geq U_2(\sigma^1, \sigma^2)$. Equivalently, in a NE (σ^1, σ^2) , σ^1 (respectively, σ^2) is a best response to σ^2 (respectively, σ^1). Let Σ denote the set of NE of Γ .

We now proceed with the equilibrium analysis of the game Γ .

5.2. Properties of Nash Equilibria

Γ is strategically equivalent to a zero-sum game; in particular, the following transformation preserves the set of NE:

$$\forall (f, I) \in \mathcal{F} \times \mathcal{I}, \quad \frac{1}{p_1} u_1(f, I) + \frac{1}{p_2} C(I) = F(f^I) - \frac{1}{p_1} T(f) + \frac{1}{p_2} C(I) =: \tilde{u}_1(f, I), \tag{22}$$

$$\forall (f, I) \in \mathcal{F} \times \mathcal{I}, \quad \frac{1}{p_2} u_2(f, I) - F(f) + \frac{1}{p_1} T(f) = -F(f^I) + \frac{1}{p_1} T(f) - \frac{1}{p_2} C(I) =: -\tilde{u}_1(f, I). \tag{23}$$

Therefore, Γ and $\tilde{\Gamma} := \langle \{1, 2\}, (\mathcal{F}, \mathcal{I}), (\tilde{u}_1, -\tilde{u}_1) \rangle$ have the same equilibrium set. Additionally, NE of Γ are interchangeable, that is, if $(\sigma^1, \sigma^2) \in \Sigma$ and $(\sigma^1, \sigma^2) \in \Sigma$, then $(\sigma^1, \sigma^2) \in \Sigma$ and $(\sigma^1, \sigma^2) \in \Sigma$. Also note that because of the splittable nature of the flow for any routing strategy $\sigma^1 \in \Delta(\mathcal{F})$ of **P1**, one can consider an equivalent pure strategy $\tilde{f} \in \mathcal{F}$ defined by $\tilde{f}_\lambda = \mathbb{E}_{\sigma^1}[f_\lambda]$ for all $\lambda \in \Lambda$, which satisfies $U_i(\sigma^1, \sigma^2) = U_i(\tilde{f}, \sigma^2)$ for all $i \in \{1, 2\}$ and $\sigma^2 \in \Delta(\mathcal{I})$, because $u_i(\cdot, I)$ is an affine function for every $I \in \mathcal{I}$.

The abovementioned properties imply that linear programming techniques can be used to obtain the NE of Γ . However, this would entail solving a linear program of exponential size, containing $|\Lambda| + 1$ variables and $2^{|\mathcal{E}|} + |\mathcal{E}|$ constraints. Instead, we derive an approach for efficiently solving Γ : We show that by utilizing the primal and dual solutions of a minimum-cost circulation problem and applying our results on posets (Theorems 1 and 2), we can obtain a complete equilibrium characterization for game Γ . Furthermore, using Algorithm 2, we obtain a polynomial-time approach to compute NE of this game.

We begin by considering the following “natural” network flow problem:

$$\begin{aligned}
 (\mathcal{M}) : \quad & \text{maximize} && F(f) - \frac{1}{p_1} T(f) \\
 & \text{subject to} && \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_\lambda \leq \min \left\{ \frac{d_{ij}}{p_2}, c_{ij} \right\}, \quad \forall (i,j) \in \mathcal{E} \\
 & && f_\lambda \geq 0, \quad \forall \lambda \in \Lambda.
 \end{aligned}$$

This problem consists in finding a feasible flow f in \mathcal{F} that maximizes $u_1(f, \emptyset)$ with the constraint that the flow through each edge (i, j) is no more than d_{ij}/p_2 . Game theoretically, this threshold captures **P2**’s best response to **P1**. Indeed, if $f_{ij} > d_{ij}/p_2$ for some $(i, j) \in \mathcal{E}$, then **P2** has an incentive to interdict (i, j) , resulting in an increase of **P2**’s payoff (because $u_2(f, \{(i, j)\}) = p_2 f_{ij} - d_{ij} > 0$). Thus, (\mathcal{M}) can be viewed as the problem in which **P1** maximizes its payoff while limiting **P2**’s incentive to interdict any of the edges. For each $s - t$ path $\lambda \in \Lambda$, let us denote $\pi_\lambda^0 := 1 - b_\lambda/p_1$. Then, the value $p_1 \pi_\lambda^0$ represents the gain in **P1**’s payoff when one unit of flow traveling along λ reaches the destination node. The primal and dual formulations of (\mathcal{M}) are given as follows:

$$\begin{array}{l|l}
 (\mathcal{M}_P) : \text{ maximize} & \sum_{\lambda \in \Lambda} \pi_\lambda^0 f_\lambda \\
 \text{subject to} & \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_\lambda \leq \frac{d_{ij}}{p_2}, \quad \forall (i,j) \in \mathcal{E} \\
 & \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_\lambda \leq c_{ij}, \quad \forall (i,j) \in \mathcal{E} \\
 & f_\lambda \geq 0, \quad \forall \lambda \in \Lambda \\
 \hline
 (\mathcal{M}_D) : \text{ minimize} & \sum_{(i,j) \in \mathcal{E}} \left(\frac{d_{ij}}{p_2} \rho_{ij} + c_{ij} \mu_{ij} \right) \\
 \text{subject to} & \sum_{(i,j) \in \lambda} (\rho_{ij} + \mu_{ij}) \geq \pi_\lambda^0, \quad \forall \lambda \in \Lambda \\
 & \rho_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{E} \\
 & \mu_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{E}.
 \end{array}$$

Let $\mathcal{O}_{(\mathcal{M}_P)}^*$ (respectively, $\mathcal{O}_{(\mathcal{M}_D)}^*$) denote the set of optimal solutions of (\mathcal{M}_P) (respectively, (\mathcal{M}_D)). By strong duality, the optimal value of (\mathcal{M}_P) is identical to that of (\mathcal{M}_D) ; we denote it by $z_{(\mathcal{M})}^*$. Note that (\mathcal{M}_P) and (\mathcal{M}_D) may have an exponential number of variables and constraints, respectively. However, equivalent polynomial-size primal and dual formulations of (\mathcal{M}) can be derived; see Appendix C. Thus, $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$ and $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$ can be efficiently computed by using an interior point method (Karmarkar [22]) or a dual network simplex algorithm (Orlin et al. [27]). Alternatively, (\mathcal{M}) can be formulated as a minimum-cost circulation problem in a graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ such that $\mathcal{V}' = \mathcal{V}$, $\mathcal{E}' = \mathcal{E} \cup \{(t, s)\}$. The capacity of each edge $(i, j) \in \mathcal{E}$ is given by $\min\{d_{ij}/p_2, c_{ij}\}$, and edge (t, s) is uncapacitated. The transportation cost of each edge $(i, j) \in \mathcal{E}$ is given by b_{ij}/p_1 , and the transportation cost of edge (t, s) is -1 . Thus, (\mathcal{M}_P) and (\mathcal{M}_D) can be solved using known combinatorial algorithms (Ahuja et al. [2]).

From complementary slackness, we know that any pair of optimal primal and dual solutions $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$ and $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$ satisfies the following properties:

$$\forall (i, j) \in \mathcal{E}, \quad \rho_{ij}^* > 0 \Rightarrow f_{ij}^* = \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_\lambda^* = \frac{d_{ij}}{p_2}, \tag{24}$$

$$\forall (i, j) \in \mathcal{E}, \quad \mu_{ij}^* > 0 \Rightarrow f_{ij}^* = \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_\lambda^* = c_{ij}, \tag{25}$$

$$\forall \lambda \in \Lambda, \quad f_\lambda^* > 0 \Rightarrow \sum_{(i,j) \in \lambda} (\rho_{ij}^* + \mu_{ij}^*) = \pi_\lambda^0. \tag{26}$$

These properties, along with Theorems 1 and 2, enable us to derive the following result:

Theorem 3. A strategy profile $(\sigma^1, \sigma^2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{I})$ is an NE of the game Γ if and only if there exists a pair of optimal primal and dual solutions $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$ and $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$ such that

$$\forall \lambda \in \Lambda, \quad \sum_{f \in \mathcal{F}} \sigma_f^1 f_\lambda = f_\lambda^*, \tag{27}$$

$$\forall (i, j) \in \mathcal{E}, \quad \sum_{\{I \in \mathcal{I} \mid (i,j) \in I\}} \sigma_I^2 = \rho_{ij}^*, \tag{28}$$

$$\forall \lambda \in \Lambda, \quad \sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \geq \pi_\lambda^*, \tag{29}$$

where $\pi_\lambda^* := \pi_\lambda^0 - \sum_{(i,j) \in \lambda} \mu_{ij}^*$ for all $\lambda \in \Lambda$. The corresponding equilibrium payoffs are $U_1(\sigma^1, \sigma^2) = p_1 \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij}^*$ and $U_2(\sigma^1, \sigma^2) = 0$.

Thus, optimal primal and dual solutions of (\mathcal{M}) provide necessary and sufficient conditions for a strategy profile to be an NE. In particular, optimal primal solutions represent the expected flows sent by **P1** in equilibrium. Additionally, optimal dual solutions characterize the marginal probabilities with which network components are interdicted in equilibrium; that is, **P2**'s equilibrium strategy interdicts each edge $(i, j) \in \mathcal{E}$ with probability ρ_{ij}^* and interdicts each path $\lambda \in \Lambda$ with probability at least $\pi_\lambda^0 - \sum_{(i,j) \in \lambda} \mu_{ij}^*$.

Although showing that (27)–(29) are sufficient conditions is relatively straightforward, the key challenge lies in proving that they are also necessary. In proving Theorem 3, we first show the existence of an interdiction strategy $\sigma^2 \in \Delta(\mathcal{I})$ satisfying (28) and (29) given an optimal dual solution of (\mathcal{M}) . In fact, this existence problem is an instantiation of Problem (\mathcal{D}) that we introduced in Section 2.2 and positively answered in Theorem 1. Secondly, showing that (28) and (29) are necessary conditions satisfied by **P2**'s interdiction strategies in equilibrium involves exploiting strong duality in the strategically equivalent zero-sum game $\tilde{\Gamma}$. Finally, the necessary Condition (27) is a consequence of the $s - t$ paths having positive transportation costs. The proof exploits Theorem 2, which guarantees the existence of **P2**'s strategy that, with positive probability $1 - \max\{\max_{(i,j) \in \mathcal{E}} \rho_{ij}^*, \max\{1 - b_\lambda/p_1 - \sum_{(i,j) \in \lambda} \mu_{ij}^*, \lambda \in \Lambda\}\}$, does not interdict any edges at all in equilibrium.

We remark that for the case when the path transportation costs are assumed to be nonnegative (instead of strictly positive), Conditions (28) and (29) are still necessary and sufficient for equilibrium interdiction strategies. However, (27) is only a sufficient condition for equilibrium routing strategies. Indeed, if a path with low interdiction costs has zero cost of transportation, **P2** will interdict this path with probability one. Any flow sent by **P1** along this path will then always be interdicted, and in fact **P1** can select an equilibrium strategy that saturates this path and violates constraints in (\mathcal{M}_p) .

In fact, dual solutions of (\mathcal{M}) can be used to infer additional equilibrium properties: Given an $s - t$ path $\lambda \in \Lambda$, π_λ^0 is the probability above which λ must be interdicted by **P2** to limit **P1**'s incentive to send any flow through the network. However, when edges belonging to λ have high interdiction costs, **P2** chooses not to interdict these edges, which may result in the interdiction probability of λ being less than π_λ^0 . This reduction of interdiction probability of λ is captured by $\sum_{(i,j) \in \lambda} \mu_{ij}^*$. By complementary slackness (25), $\mu_{ij}^* > 0$ for $(i, j) \in \lambda$ only when $c_{ij} = f_{ij}^* \leq d_{ij}/p_2$. Hence, the equilibrium interdiction probability of λ is given by $\pi_\lambda^* = \pi_\lambda^0 - \sum_{(i,j) \in \lambda} \mu_{ij}^*$.

Consequently, if an $s - t$ path $\lambda \in \Lambda$ is such that $\sum_{(i,j) \in \lambda} \mu_{ij}^* > 0$, then each unit of flow sent through λ increases **P1**'s payoff by $p_1 \sum_{(i,j) \in \lambda} \mu_{ij}^*$. This is captured by **P1**'s equilibrium strategies, with expected flow $f^* \in \mathcal{O}^*(\mathcal{M}_p)$, that saturate every edge $(i, j) \in \mathcal{E}$ for which $\mu_{ij}^* > 0$. Since f^* only takes $s - t$ paths that are interdicted with probability exactly π^* , the resulting equilibrium payoff for **P1** is given by $p_1 \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij}^*$. Note also that f^* does not take any $s - t$ path λ for which $\pi_\lambda^0 < 0$. This captures the fact that **P1** has no incentive to send its flow through $s - t$ paths λ for which $b_\lambda > p_1$. Recall from (\mathcal{M}) that f^* is such that interdicting any edge does not increase **P2**'s payoff. Furthermore, **P2** only interdicts edges for which the value from the interdicted flow compensates the interdiction cost (from (24)). Thus, **P2**'s payoff is zero in equilibrium. It is interesting to note that **P1**'s strategies and payoff in equilibrium can be expressed in terms of edge values and are independent of the chosen path decomposition of f^* .

Proof of Theorem 3. We prove this theorem by showing that Conditions (27)–(29) are sufficient for a strategy profile to be an NE (Step 1); satisfied by at least one strategy profile (Step 2); and satisfied by every NE (Step 3).

Step 1. Let $f^* \in \mathcal{O}^*(\mathcal{M}_p)$ and $(\rho^*, \mu^*) \in \mathcal{O}^*(\mathcal{M}_D)$. First, we show that a strategy profile $(\sigma^1, \sigma^2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{I})$ satisfying (27)–(29) is an NE of Γ . We write the following inequality for **P1**'s payoff:

$$\begin{aligned} \forall f \in \mathcal{F}, U_1(f, \sigma^2) &\stackrel{(20)}{=} p_1 \sum_{\lambda \in \Lambda} f_\lambda \mathbb{E}_{\sigma^2} [1 - \mathbb{1}_{\{I \cap \lambda \neq \emptyset\}}] - \sum_{\lambda \in \Lambda} b_\lambda f_\lambda = p_1 \sum_{\lambda \in \Lambda} \pi_\lambda^0 f_\lambda - p_1 \sum_{\lambda \in \Lambda} f_\lambda \sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \\ &\stackrel{(29)}{\leq} p_1 \sum_{\lambda \in \Lambda} f_\lambda \sum_{(i,j) \in \lambda} \mu_{ij}^* = p_1 \sum_{(i,j) \in \mathcal{E}} f_{ij} \mu_{ij}^* \leq p_1 \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij}^*. \end{aligned} \tag{30}$$

Now, given $\lambda \in \Lambda$ such that $f_\lambda^* > 0$, we obtain

$$\sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \leq \sum_{I \in \mathcal{I}} \sigma_I^2 |I \cap \lambda| = \sum_{(i,j) \in \lambda} \sum_{I \in \mathcal{I}} \sigma_I^2 \mathbb{1}_{\{(i,j) \in I\}} \stackrel{(28)}{=} \sum_{(i,j) \in \lambda} \rho_{ij}^* \stackrel{(26), (29)}{\leq} \sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2. \tag{31}$$

Furthermore, for all $(i, j) \in \mathcal{E}$ such that $\mu_{ij}^* > 0$, $f_{ij}^* \stackrel{(25)}{=} c_{ij}$. Then, Inequality (30) is tight for f^* , and $U_1(\sigma^1, \sigma^2) \stackrel{(27)}{=} U_1(f^*, \sigma^2) = p_1 \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij}^*$.

Similarly, regarding **P2**'s payoff, we first derive the following inequality:

$$\forall I \in \mathcal{I}, \sum_{(i,j) \in I} \frac{d_{ij}}{p_2} \geq \sum_{(i,j) \in I} \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_{\lambda}^* = \sum_{\lambda \in \Lambda} f_{\lambda}^* |I \cap \lambda| \geq \sum_{\lambda \in \Lambda} f_{\lambda}^* \mathbb{1}_{\{I \cap \lambda \neq \emptyset\}} = F(f^*) - F(f^{*I}). \quad (32)$$

Therefore, for all $I \in \mathcal{I}$, $U_2(\sigma^1, I) \stackrel{(27)}{=} U_2(f^*, I) \stackrel{(21)}{=} p_2(F(f^*) - F(f^{*I})) - \sum_{(i,j) \in I} d_{ij} \stackrel{(32)}{\leq} 0$.

Now, consider $I \in \text{supp}(\sigma^2)$. From (31), we obtain that for every $\lambda \in \Lambda$ such that $f_{\lambda}^* > 0$, $|I \cap \lambda| \leq 1$. Furthermore, for every $(i, j) \in I$, $\sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_{\lambda}^* \stackrel{(24)}{=} d_{ij}/p_2$ because $\rho_{ij}^* > 0$. Thus, for all $S \in \text{supp}(\sigma^2)$, Inequality (32) is tight, and $U_2(\sigma^1, I) = 0$. Therefore, $U_2(\sigma^1, \sigma^2) = 0$, and (σ^1, σ^2) is an NE.

Step 2. Let $f^* \in \mathcal{O}_{(\mathcal{M}_P)}$ and $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}$. Next, we show that there exists a strategy profile satisfying (27)–(29), and obtain the value of the zero-sum game $\tilde{\Gamma}$. Trivially, if **P1** chooses the pure strategy f^* , (27) is then satisfied. We now argue that there exists an interdiction strategy $\tilde{\sigma}^2 \in \Delta(\mathcal{I})$ satisfying (28) and (29). First, we define the following binary relation on \mathcal{E} , denoted $\preceq_{\mathcal{G}}$: Given $(u, v) \in \mathcal{E}^2$, $u \preceq_{\mathcal{G}} v$ if either $u = v$, or there exists an $s-t$ path $\lambda \in \Lambda$ that traverses u and v in this order. Since \mathcal{G} is a directed acyclic connected graph, we obtain the following lemma, which is proven in Appendix A:

Lemma 4. $P_{\mathcal{G}} = (\mathcal{E}, \preceq_{\mathcal{G}})$ is a poset, whose set of maximal chains is the set of $s-t$ paths Λ .

Thus, showing that there exists $\tilde{\sigma}^2 \in \Delta(\mathcal{I})$ satisfying (28) and (29) is an instantiation of Problem (D). Since $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}$, then Condition (2) is satisfied, that is, for all $\lambda \in \Lambda$, $\sum_{(i,j) \in \lambda} \rho_{ij}^* \geq \pi_{\lambda}^*$. Additionally, for any $s-t$ path $\lambda \in \Lambda$, $\pi_{\lambda}^* = 1 - \sum_{(i,j) \in \lambda} (b_{ij}/p_1 + \mu_{ij}^*)$, and π^* is an affine function of the edges constituting each $s-t$ path. Therefore, π^* satisfies the conservation law described in (3). Finally, since $\rho_{ij}^* \in [0, 1]$ for all $(i, j) \in \mathcal{E}$, and $\pi_{\lambda}^* \leq 1$ for all $\lambda \in \Lambda$, all conditions of Theorem 1 are satisfied, and there exists an interdiction strategy $\tilde{\sigma}^2 \in \Delta(\mathcal{I})$ satisfying (28) and (29). In particular, $\tilde{\sigma}^2$ can be constructed from Algorithm 2.

From Step 1, $(f^*, \tilde{\sigma}^2) \in \Sigma$. Then, **P1**'s equilibrium payoff in the zero-sum game $\tilde{\Gamma}$ is

$$\forall (\sigma^1, \sigma^2) \in \Sigma, \tilde{U}_1(\sigma^1, \sigma^2) = \tilde{U}_1(f^*, \tilde{\sigma}^2) \stackrel{(22)}{=} \frac{1}{p_1} U_1(f^*, \tilde{\sigma}^2) + \frac{1}{p_2} \mathbb{E}_{\tilde{\sigma}^2}[C(I)] \stackrel{(28)}{=} z_{(\mathcal{M})}^*. \quad (33)$$

Step 3. Let $(\sigma^1, \sigma^2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{I})$ be an NE of Γ . We now show that (σ^1, σ^2) satisfies (27)–(29) for some pair of optimal primal and dual solutions of (\mathcal{M}) . In particular, we first prove that $f' := \mathbb{E}_{\sigma^1}[f]$ is necessarily an optimal solution of (\mathcal{M}_P) : Given $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}$, consider the equilibrium interdiction strategy $\tilde{\sigma}^2$ described in Step 2 and constructed from Algorithm 2. From Theorem 2, $\tilde{\sigma}_0^2 = 1 - \max\{\max\{\rho_{ij}^*, (i, j) \in \mathcal{E}\}, \max\{\pi_{\lambda}^*, \lambda \in \Lambda\}\} > 0$. Since $\emptyset \in \text{supp}(\tilde{\sigma}^2)$ and $(\sigma^1, \tilde{\sigma}^2) \in \Sigma$, then for all $(i, j) \in \mathcal{E}$, $0 \leq U_2(\sigma^1, \emptyset) - U_2(\sigma^1, \{(i, j)\}) \stackrel{(21)}{=} d_{ij} - p_2 f'_{ij}$. Therefore, f' is a feasible solution of (\mathcal{M}_D) . In addition, $z_{(\mathcal{M})}^* \stackrel{(33)}{=} \tilde{U}_1(\sigma^1, \emptyset) \stackrel{(22)}{=} F(f') - \frac{1}{p_1} T(f')$. Thus, $f' \in \mathcal{O}_{(\mathcal{M}_P)}$.

Secondly, we show that σ^2 necessarily satisfies (28) and (29) for some optimal solution of (\mathcal{M}_D) . For every $(i, j) \in \mathcal{E}$, let $\rho'_{ij} := \sum_{\{I \in \mathcal{I} \mid (i,j) \in I\}} \sigma_I^2$. We can then derive the following inequalities:

$$\begin{aligned} z_{(\mathcal{M})}^* &\stackrel{(33)}{=} \max_{f \in \mathcal{F}} \tilde{U}_1(f, \sigma^2) \stackrel{(22)}{=} \sum_{(i,j) \in \mathcal{E}} \frac{d_{ij}}{p_2} \rho'_{ij} + \max_{f \in \mathcal{F}} \left\{ \sum_{\lambda \in \Lambda} f_{\lambda} \left(\pi_{\lambda}^0 - \sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \right) \right\} \\ &= \sum_{(i,j) \in \mathcal{E}} \frac{d_{ij}}{p_2} \rho'_{ij} + \min_{\mu \in \mathbb{R}_{\geq 0}^{\mathcal{E}}} \left\{ \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij} \mid \forall \lambda \in \Lambda, \sum_{(i,j) \in \lambda} \mu_{ij} \geq \pi_{\lambda}^0 - \sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \right\} \end{aligned} \quad (34)$$

$$\geq \sum_{(i,j) \in \mathcal{E}} \frac{d_{ij}}{p_2} \rho'_{ij} + \min_{\mu \in \mathbb{R}_{\geq 0}^{\mathcal{E}}} \left\{ \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij} \mid \forall \lambda \in \Lambda, \sum_{(i,j) \in \lambda} \mu_{ij} \geq \pi_{\lambda}^0 - \sum_{(i,j) \in \lambda} \rho'_{ij} \right\} \quad (35)$$

$$\geq \min_{\rho, \mu \in \mathbb{R}_{\geq 0}^{\mathcal{E}}} \left\{ \sum_{(i,j) \in \mathcal{E}} \left(\frac{d_{ij}}{p_2} \rho_{ij} + c_{ij} \mu_{ij} \right) \mid \forall \lambda \in \Lambda, \sum_{(i,j) \in \lambda} (\rho_{ij} + \mu_{ij}) \geq \pi_{\lambda}^0 \right\} = z_{(\mathcal{M})}^*. \quad (36)$$

Thus, inequalities (35) and (36) are tight. Note that (34) is a consequence of the strong duality theorem, and (35) holds because $\sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \leq \sum_{(i,j) \in \Lambda} \rho'_{ij}$ for every $\lambda \in \Lambda$.

Let $\mu' := \arg \min_{\mu \in \mathbb{R}_{\geq 0}^{\mathcal{E}}} \{ \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij} \mid \forall \lambda \in \Lambda, \sum_{(i,j) \in \lambda} \mu_{ij} \geq \pi_\lambda^0 - \sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \}$. Then, (35) and (36) imply that $(\rho', \mu') \in \mathcal{O}_{(\mathcal{M}_D)}^*$. Furthermore, by definition of μ' , $\sum_{\{I \in \mathcal{I} \mid I \cap \lambda \neq \emptyset\}} \sigma_I^2 \geq \pi_\lambda^0 - \sum_{(i,j) \in \lambda} \mu'_{ij}$ for every $\lambda \in \Lambda$. Thus, σ^2 satisfies (28) and (29) with $(\rho', \mu') \in \mathcal{O}_{(\mathcal{M}_D)}^*$. \square

A direct consequence of Theorem 3 is that some quantities related to the players' actions in equilibrium can be computed in closed form using the game parameters and the optimal primal and dual solutions of (\mathcal{M}) . They are summarized in the following corollary:

Corollary 1. NE of the game Γ satisfy the following properties:

- i. Expected amount (respectively, cost) of initial flow sent by **P1** is given by $F(f^*)$ (respectively, $T(f^*)$),
 - ii. Expected cost of **P2**'s interdiction strategy is given by $\sum_{(i,j) \in \mathcal{E}} d_{ij} \rho_{ij}^*$,
 - iii. Expected amount of interdicted flow is given by $\sum_{(i,j) \in \mathcal{E}} \rho_{ij}^* d_{ij} / p_2$,
 - iv. Expected amount of effective flow is given by $F(f^*) - \sum_{(i,j) \in \mathcal{E}} \rho_{ij}^* d_{ij} / p_2$,
- where $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$ and $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$.

Given $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$ and $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$, the expected amount of interdicted flow achievable by any interdiction strategy satisfying (28) is upper bounded by $\sum_{(i,j) \in \mathcal{E}} \rho_{ij}^* d_{ij} / p_2$. Property (iii) in Corollary 1 shows that this upper bound is achieved by **P2**'s strategy in equilibrium. In other words, given the marginal edge interdiction probabilities ρ^* , **P2** randomizes its interdictions to maximize the amount of interdicted flow, while still limiting **P1**'s incentive to deviate from its strategy.

Note that despite the exponential number of actions of both players, an NE can be computed in polynomial time. Indeed, we first solve the polynomial-size formulation of (\mathcal{M}) and use the flow decomposition algorithm to obtain $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$ and $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$ (see Appendix C). Since π^* is an affine function of the edges constituting each $s-t$ path, we run Algorithm 2 on the poset $P_G = (\mathcal{E}, \preceq_G)$ (Lemma 4) to compute an interdiction strategy $\tilde{\sigma}^2 \in \Delta(\mathcal{I})$ satisfying (28) and (29). Given $H_{P_G} = (\mathcal{E}, E_{P_G})$ the directed cover graph of P_G , we deduce that $\tilde{\sigma}^2$ can be obtained in $O(|\mathcal{E}| |E_{P_G}|^2)$ time. Since \mathcal{G} is a simple directed acyclic graph, then each $(i, j) \in \mathcal{E}$ is adjacent to at most $|\mathcal{V}| - 2$ edges (i', j') in \mathcal{G} such that $i' = j$ or $j' = i$. This in turn implies that the degree of each $(i, j) \in \mathcal{E}$ in H_{P_G} is at most $|\mathcal{V}| - 2$. Therefore, the total number of edges in H_{P_G} is upper bounded by $|E_{P_G}| \leq \frac{1}{2} |\mathcal{E}| (|\mathcal{V}| - 2)$. In conclusion, the NE $(f^*, \tilde{\sigma}^2)$ is computed in $O(|\mathcal{V}|^2 |\mathcal{E}|^3)$ time. In this NE, **P1** sends its flow along at most $|\mathcal{E}| s-t$ paths (from the flow decomposition theorem), and **P2** randomizes over at most $\frac{1}{2} |\mathcal{V}| |\mathcal{E}| + 1$ interdictions (given by the number of iterations in Algorithm 2).

We remark that in the simpler case where each $s-t$ path has an identical transportation cost, (\mathcal{M}) can be viewed as a maximum flow problem. Then, this solution approach simply computes an NE of Γ from a maximum flow for **P1** and a minimum-cut set for **P2**.

For the sake of completeness, we characterize the game instances for which pure NE exist. From Theorem 3, a pure NE exists if and only if there exists $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$ such that $\rho^* \in \{0, 1\}^{\mathcal{E}}$. Since $b_\lambda > 0$ for every $\lambda \in \Lambda$, then $\rho_{ij}^* < 1$ for every $(i, j) \in \mathcal{E}$ at optimality of (\mathcal{M}_D) , and a pure NE exists if and only if $\rho_{ij}^* = 0$ for every $(i, j) \in \mathcal{E}$. The corresponding pure NE are (f^*, \emptyset) with $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$. Note that this case occurs when the interdiction costs for **P2** or transportation costs for **P1** are too high.

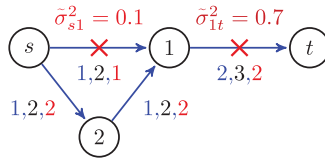
Finally, we analyze the set of $s-t$ paths (respectively, set of edges) that are chosen (respectively, interdicted) in at least one NE. From Theorem 3, the set of $s-t$ paths chosen by **P1** in at least one NE is given by $\cup_{f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*} \text{supp}(f^*)$. Similarly, the set of edges interdicted by **P2** in at least one NE is given by $\cup_{(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*} \text{supp}(\rho^*)$. To efficiently compute these sets of critical components, we utilize the notion of *strict complementary slackness*. Specifically, optimal solutions f^\dagger and $(\rho^\dagger, \mu^\dagger)$ of (\mathcal{M}_P) and (\mathcal{M}_D) satisfy strict complementary slackness if

$$\forall (i, j) \in \mathcal{E}, \text{ either } \rho_{ij}^\dagger > 0 \text{ or } f_{ij}^\dagger = \sum_{\{\lambda \in \Lambda \mid (i, j) \in \lambda\}} f_\lambda^\dagger < \frac{d_{ij}}{p_2}, \quad (37)$$

$$\forall (i, j) \in \mathcal{E}, \text{ either } \mu_{ij}^\dagger > 0 \text{ or } f_{ij}^\dagger = \sum_{\{\lambda \in \Lambda \mid (i, j) \in \lambda\}} f_\lambda^\dagger < c_{ij}, \quad (38)$$

$$\forall \lambda \in \Lambda, \text{ either } f_\lambda^\dagger > 0 \text{ or } \sum_{(i, j) \in \lambda} (\rho_{ij}^\dagger + \mu_{ij}^\dagger) > \pi_\lambda^0. \quad (39)$$

Figure 5. (Color online) NE when $p_1 = 10, p_2 = 1, b_{ij} = 1$ for all $(i, j) \in \mathcal{E}$. The label of each edge (i, j) represents $(f_{ij}^\dagger, c_{ij}, d_{ij})$. Edge $(s, 1)$ is interdicted by the equilibrium interdiction strategy $\tilde{\sigma}^2$, but is not part of the minimum-cut set.



We say that f^\dagger and $(\rho^\dagger, \mu^\dagger)$ form a *strictly complementary primal-dual pair* of optimal solutions of (\mathcal{M}) . Note that such a pair is guaranteed to exist by the Goldman-Tucker theorem (Goldman and Tucker [16]). We now show the following result:

Proposition 6. *Let f^\dagger and $(\rho^\dagger, \mu^\dagger)$ be a strictly complementary primal-dual pair of optimal solutions of (\mathcal{M}) . The set of $s - t$ paths (respectively, the set of edges) chosen with positive probability by **P1**'s strategy (respectively, **P2**'s strategy) in at least one NE is given by $\text{supp}(f^\dagger)$ (respectively, $\text{supp}(\rho^\dagger)$).*

Proof of Proposition 6. Let f^\dagger and $(\rho^\dagger, \mu^\dagger)$ be optimal solutions of (\mathcal{M}_P) and (\mathcal{M}_D) that satisfy strict complementary slackness. We denote $\tilde{\sigma}^2 \in \Delta(\mathcal{I})$ the interdiction strategy, constructed from Algorithm 2, that interdicts every edge $(i, j) \in \mathcal{E}$ with probability ρ_{ij}^\dagger , and interdicts every $s - t$ path $\lambda \in \Lambda$ with probability at least $\pi_\lambda^0 - \sum_{(i,j) \in \lambda} \mu_{ij}^\dagger$. Given Σ the set of NE of the game Γ , let $\mathcal{H}_1 := \cup_{(\sigma^1, \sigma^2) \in \Sigma} \cup_{f \in \text{supp}(\sigma^1)} \text{supp}(f)$ and $\mathcal{H}_2 := \cup_{(\sigma^1, \sigma^2) \in \Sigma} \cup_{I \in \text{supp}(\sigma^2)} I$.

From Theorem 3, we know that $(f^\dagger, \tilde{\sigma}^2) \in \Sigma$. Consequently, $\mathcal{H}_1 \supseteq \text{supp}(f^\dagger)$ and $\mathcal{H}_2 \supseteq \text{supp}(\rho^\dagger)$. To show the reverse inclusions, consider $(\sigma^1, \sigma^2) \in \Sigma$. Theorem 3 implies that there exists $(\rho^*, \mu^*) \in \mathcal{O}_{(\mathcal{M}_D)}^*$ such that $\sum_{\{I \in \mathcal{I} \mid (i,j) \in I\}} \sigma_I^2 = \rho_{ij}^*$. Consider $(i, j) \in \mathcal{E}$ such that $\rho_{ij}^* > 0$. By complementary slackness between (ρ^*, μ^*) and f^\dagger , $d_{ij}/p_2 \stackrel{(24)}{=} \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f_\lambda^\dagger$. Then, from strict complementary slackness (37), $\rho_{ij}^\dagger > 0$. Therefore, $\mathcal{H}_2 \subseteq \text{supp}(\rho^\dagger)$, which implies that $\mathcal{H}_2 = \text{supp}(\rho^\dagger)$.

Similarly, given $(\sigma^1, \sigma^2) \in \Sigma$, Theorem 3 implies that there exists $f^* \in \mathcal{O}_{(\mathcal{M}_P)}^*$ such that $f^* = \mathbb{E}_{\sigma^1}[f]$. Consider $\lambda \in \Lambda$ such that $f_\lambda^* > 0$. Then, by complementary slackness between f^* and $(\rho^\dagger, \mu^\dagger)$, $\sum_{(i,j) \in \lambda} (\rho_{ij}^\dagger + \mu_{ij}^\dagger) \stackrel{(26)}{=} \pi_\lambda^0$. From (39), $f_\lambda^\dagger > 0$. Therefore, $\mathcal{H}_1 \subseteq \text{supp}(f^\dagger)$; we conclude that $\mathcal{H}_1 = \text{supp}(f^\dagger)$. \square

Thus, by computing a strictly complementary primal-dual pair of optimal solutions f^\dagger and $(\rho^\dagger, \mu^\dagger)$ of (\mathcal{M}) , Proposition 6 shows that the set of critical $s - t$ paths of the network is given by $\text{supp}(f^\dagger)$, and the set of critical network edges is given by $\text{supp}(\rho^\dagger)$. Such a pair can be efficiently computed using any of the existing methods in the literature (see Adler and Monteiro [1], Balinski and Tucker [7], Jansen et al. [21]).

We note that in the setting that we consider, **P2** may need to interdict edges that are not part of any minimum-cut set and can even belong to different cut sets; Figure 5 illustrates an example. In this example, the equilibrium interdiction strategy targets edges $(s, 1)$ and $(1, t)$ that do not belong to a same cut set. Thus, Proposition 6 generalizes the previously studied max-flow min-cut-based metrics of network criticality (see Assadi et al. [4], Dwivedi and Yu [14], Gueye et al. [18]).

In summary, our results in Section 5 provide a new approach to solve the strategic interdiction game Γ and derive equilibrium properties for settings involving multiple interdictions, heterogeneous cost parameters, and general network topology.

6. Concluding Remarks

In this article, we studied an existence problem of probability distributions over partially ordered sets and showed its applications to a class of interdiction games on flow networks. In the existence problem, we considered a poset, where each element and each maximal chain is associated with a value. Under two relevant conditions on these values, we showed that there exists a probability distribution over the subsets of this poset, with the following properties: the probability that each element (respectively, maximal chain) is contained in a subset (respectively, intersects with a subset) is equal to (respectively, as large as) the corresponding value. We provided a constructive proof of this result by designing a combinatorial algorithm that exploits

structural properties of the problem. In the special case where the maximal chain values depend affinely on their constituting elements, we refined our algorithm to compute a probability distribution that satisfies the desired properties in polynomial time.

By applying this existence result, we solved a generic formulation of strategic network interdiction game between a routing entity and an interdictor. To overcome the computational and analytical challenges of the formulation, we proposed a new approach for characterizing equilibria of the game. This approach relies on our existence result on posets as well as optimal primal and dual solutions of a minimum-cost circulation problem. In addition, we showed that Nash equilibria of the game can be efficiently computed with our refined algorithm on posets. Finally, we demonstrated that the critical network components that are chosen in equilibrium by both players can be computed from a strictly complementary primal-dual pair of optimal solutions of the circulation problem.

Acknowledgments

The authors are grateful to Bernhard von Stengel, the associate editor, and the reviewers for useful suggestions that improved the quality of this article.

Appendix A. Remaining Proofs

Proof of Lemma 1. Let P be a finite nonempty poset, and let S be the set of minimal elements of P . If $|S|=1$, then S is an antichain of P . Now, assume that $|S| \geq 2$, and consider $(x, y) \in S^2$ with $x \neq y$. Since x (respectively, y) is a minimal element of P , then $y \not\prec x$ (respectively, $x \not\prec y$). Therefore, x and y are incomparable, and S is an antichain of P .

Now, consider a maximal chain $C \in \mathcal{C}$ and assume that C does not contain any minimal element of P . Let x be the minimal element of $(C, \preceq|_C)$. Since x is not a minimal element of P , there exists $y \in X \setminus C$ such that $y \prec x$. By transitivity of \preceq , $y \prec x'$ for every $x' \in C$. Therefore, $C \cup \{y\}$ is a chain containing C , which contradicts the maximality of C . Thus, every maximal chain of P intersects with the set of minimal elements of P . \square

Proof of Lemma 2. Consider $X' \subseteq X$ and $\mathcal{C}' \subseteq \mathcal{C}$ that preserves the decomposition of maximal chains intersecting in X' . Let us show that $\preceq_{\mathcal{C}'}$ defined in Section 2.1 is a partial order on X' :

- Reflexivity: For every $x \in X'$, $x \preceq_{\mathcal{C}'} x$ by definition.
- Antisymmetry: Consider $(x, y) \in (X')^2$ such that $x \preceq_{\mathcal{C}'} y$ and $y \preceq_{\mathcal{C}'} x$. If $x \neq y$, then we would have $x \prec y$ and $y \prec x$, which contradicts \preceq being a partial order. Therefore, $x = y$.
- Transitivity: Consider $(x, y, z) \in (X')^3$, and assume that $x \preceq_{\mathcal{C}'} y$ and $y \preceq_{\mathcal{C}'} z$. If $x = y$ or $y = z$, then we trivially obtain that $x \preceq_{\mathcal{C}'} z$. Now, let us assume that $x \neq y$ and $y \neq z$. By definition of $\preceq_{\mathcal{C}'}$, there exists $C^1 \in \mathcal{C}'$ such that $(x, y) \in (C^1)^2$ and $x \prec y$. Similarly, there exists $C^2 \in \mathcal{C}'$ such that $(y, z) \in (C^2)^2$ and $y \prec z$. We can rewrite C^1 and C^2 as follows: $C^1 = \{x_0, \dots, x_l = x, x_{l+1}, \dots, x_{l+m} = y, x_{l+m+1}, \dots, x_{l+m+n}\}$ and $C^2 = \{y_0, \dots, y_q = y, y_{q+1}, \dots, y_{q+r} = z, y_{q+r+1}, \dots, y_{q+r+s}\}$. Now, consider the maximal chain $C_1^2 = \{x_0, \dots, x_l = x, x_{l+1}, \dots, x_{l+m} = y, y_{q+1}, \dots, y_{q+r} = z, y_{q+r+1}, \dots, y_{q+r+s}\}$, as illustrated in Figure A.1.

Since C^1 and C^2 intersect in $y \in X'$, and \mathcal{C}' preserves the decomposition of maximal chains intersecting in X' , we deduce that $C_1^2 \in \mathcal{C}'$ as well. Furthermore, $(x, z) \in (C_1^2)^2$, and the transitivity of \preceq implies that $x \prec z$. Therefore, $x \preceq_{\mathcal{C}'} z$.

Thus, $\preceq_{\mathcal{C}'}$ is a partial order on X' , and $P' = (X', \preceq_{\mathcal{C}'})$ is a poset.

Let $C \subseteq X'$ be a maximal chain of P' of size at least two. Let us rewrite $C = \{x_1, \dots, x_n\}$ with $n \geq 2$, where for all $k \in \llbracket 1, n-1 \rrbracket$, $x_k \prec_{\mathcal{C}'} x_{k+1}$. We show by induction on $k \in \llbracket 2, n \rrbracket$ that there exists $C' \in \mathcal{C}'$ such that $\{x_1, \dots, x_k\} \subseteq C'$. If $k = 2$,

Figure A.1. (Color online) Illustration of the transitivity of $\preceq_{\mathcal{C}'}$. C_1^2 is represented by the thick chain.

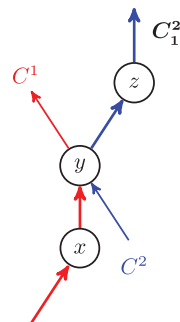
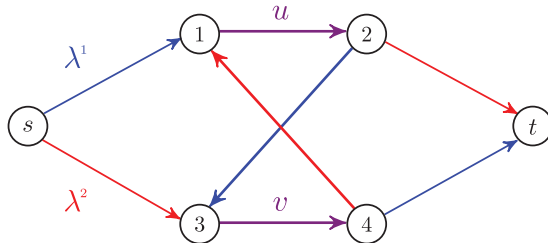


Figure A.2. (Color online) Proof of antisymmetry of $\preceq_{\mathcal{G}}$ by contradiction: if $v \preceq_{\mathcal{G}} v, v \preceq_{\mathcal{G}} u$, and $u \neq v$, then one can see that u and v necessarily belong to a cycle (shown in thick edges), although \mathcal{G} is acyclic.



then by definition, there exists $C' \in \mathcal{C}'$ such that $\{x_1, x_2\} \subseteq C'$. Now, assume that the result holds for $k \in \llbracket 2, n-1 \rrbracket$. Consider $C^1 \in \mathcal{C}'$ such that $\{x_1, \dots, x_k\} \subseteq C^1$. Since $x_k \prec_{\mathcal{C}'} x_{k+1}$, then there exists $C^2 \in \mathcal{C}'$ such that $(x_k, x_{k+1}) \in (C^2)^2$. Analogously, we can show that C_1^2 (illustrated in Figure A.1), which is in \mathcal{C}' , contains $\{x_1, \dots, x_{k+1}\}$. Therefore, by induction, there exists $C' \in \mathcal{C}'$ such that $C = \{x_1, \dots, x_n\} \subseteq C'$. Since $C \subseteq X'$, then we have $C = C \cap X' \subseteq C' \cap X'$.

Now, assume that there exists $x' \in C' \cap X' \setminus C$. For every $k \in \llbracket 1, n \rrbracket$, $(x_k, x') \in (C')^2$. Therefore, x' is comparable in P' with every element of the chain C . This implies that $C \cup \{x'\}$ is a chain in P' , which contradicts the maximality of C in P' . Therefore, $C = C' \cap X'$. \square

Proof of Lemma 4. Let us show that $\preceq_{\mathcal{G}}$ is a partial order on \mathcal{E} .

– Reflexivity: For every $u \in \mathcal{E}$, $u \preceq_{\mathcal{G}} u$ by definition.

– Antisymmetry: Consider $(u, v) \in \mathcal{E}^2$ such that $u \preceq_{\mathcal{G}} v$ and $v \preceq_{\mathcal{G}} u$. If $u \neq v$, then there exist λ^1 and λ^2 in Λ such that λ^1 traverses u and v in this order, and λ^2 traverses v and u in this order. They can be written as follows: $\lambda^1 = \{u_1, \dots, u_n, u, u_{n+1}, \dots, u_{n+m}, v, u_{n+m+1}, \dots, u_{n+m+p}\}$ and $\lambda^2 = \{v_1, \dots, v_q, v, v_{q+1}, \dots, v_{q+r}, u, v_{q+r+1}, \dots, v_{q+r+s}\}$. Then, $\{u, u_{n+1}, \dots, u_{n+m}, v, v_{q+1}, \dots, v_{q+r}\}$ is a cycle (see Figure A.2), which contradicts \mathcal{G} being acyclic. Therefore, $u = v$.

– Transitivity: Consider $(u, v, w) \in \mathcal{E}^3$ and assume that $u \preceq_{\mathcal{G}} v$ and $v \preceq_{\mathcal{G}} w$. If $u = v$ or $v = w$, then we trivially obtain that $u \preceq_{\mathcal{G}} w$. Now, let us assume that $u \neq v$ and $v \neq w$. Then, there exist λ^1 and λ^2 in Λ such that λ^1 traverses u and v in this order and λ^2 traverses v and w in this order. They can be written as $\lambda^1 = \{u_1, \dots, u_n, u, u_{n+1}, \dots, u_{n+m}, v, u_{n+m+1}, \dots, u_{n+m+p}\}$ and $\lambda^2 = \{v_1, \dots, v_q, v, v_{q+1}, \dots, v_{q+r}, w, v_{q+r+1}, \dots, v_{q+r+s}\}$. Then, $\lambda_1^2 = \{u_1, \dots, u_n, u, u_{n+1}, \dots, u_{n+m}, v, v_{q+1}, \dots, v_{q+r}, w, v_{q+r+1}, \dots, v_{q+r+s}\}$ is an $s-t$ path (see Figure A.3) and traverses u and w in this order. Therefore, $u \preceq_{\mathcal{G}} w$.

In conclusion, $P_{\mathcal{G}} = (\mathcal{E}, \preceq_{\mathcal{G}})$ is a poset.

Next, we prove that the set of maximal chains \mathcal{C} of $P_{\mathcal{G}}$ is Λ . Consider a maximal chain $C \in \mathcal{C}$ of $P_{\mathcal{G}}$. If $C = \{u\}$ is of size one, then necessarily $u = (s, t)$, because \mathcal{G} is connected. Therefore, $C = \{u\}$ is an $s-t$ path. Now, assume that $|C| \geq 2$. Let us write $C = \{u_1, \dots, u_n\}$, where for every $k \in \llbracket 1, n-1 \rrbracket$, $u_k \prec_{\mathcal{G}} u_{k+1}$. Since $u_1 \prec_{\mathcal{G}} u_2$ and $u_2 \prec_{\mathcal{G}} u_3$, then there exist λ^1 and λ^2 in Λ such that λ^1 traverses u_1 and u_2 in this order, and λ^2 traverses u_2 and u_3 in this order. When proving the transitivity of $\preceq_{\mathcal{G}}$, we showed that there exists $\lambda_1^2 \in \Lambda$ that traverses u_1, u_2 , and u_3 in this order. By repeating this process, we obtain an $s-t$ path $\lambda \in \Lambda$ such that $C \subseteq \lambda$.

Now, assume that there exists $u \in \lambda \setminus C$. Since $C \subseteq \lambda$ and $u \in \lambda$, then u is comparable with every element of C (by definition of $\preceq_{\mathcal{G}}$). Therefore, $C \cup \{u\}$ is a chain in $P_{\mathcal{G}}$, which contradicts the maximality of C . Therefore, $C = \lambda$ and $C \subseteq \Lambda$.

To show the reverse inclusion, consider an $s-t$ path $\lambda \in \Lambda$. By definition of $\preceq_{\mathcal{G}}$, λ is a chain in $P_{\mathcal{G}}$. Let us assume that there exists a maximal chain $C \in \mathcal{C}$ such that $\lambda \not\subseteq C$. Let us write $\lambda = \{u_1, \dots, u_n\}$ where for every $k \in \llbracket 1, n-1 \rrbracket$, $u_k \prec_{\mathcal{G}} u_{k+1}$, and let $v \in C \setminus \lambda$. Since $\lambda \subset C$ and $v \in C$, then v is comparable with every element of λ . By transitivity of $\preceq_{\mathcal{G}}$, if there exists $k \in \llbracket 1, n \rrbracket$ such that $v \prec_{\mathcal{G}} u_k$, then for every $l \in \llbracket k, n \rrbracket$, $v \prec_{\mathcal{G}} u_l$. Similarly, if there exists $k \in \llbracket 1, n \rrbracket$ such that $u_k \prec_{\mathcal{G}} v$, then for every $l \in \llbracket 1, k \rrbracket$, $u_l \prec_{\mathcal{G}} v$. Therefore, three cases can arise:

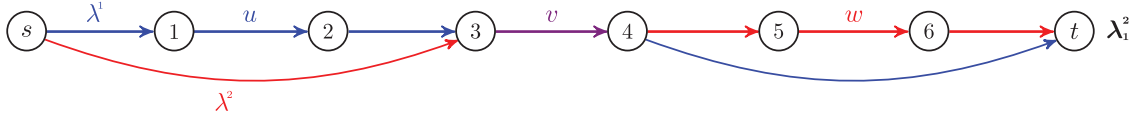
– $v \prec_{\mathcal{G}} u_1$. In this case, there exists $\lambda^1 = \{w_1, \dots, w_n, v, w_{n+1}, \dots, w_{n+m}, u_1, w_{n+m+1}, \dots, w_{n+m+p}\} \in \Lambda$. However, since λ is an $s-t$ path, then the start node of u_1 is s , which is also the start node of w_1 . Therefore, $\{w_1, \dots, w_n, v, w_{n+1}, \dots, w_{n+m}\}$ is a cycle, which is a contradiction.

– $u_n \prec_{\mathcal{G}} v$. In this case, there exists $\lambda^1 = \{v_1, \dots, v_q, u_n, v_{q+1}, \dots, v_{q+r}, v, v_{q+r+1}, \dots, v_{q+r+s}\} \in \Lambda$. Analogously, $\{v_{q+1}, \dots, v_{q+r}, v, v_{q+r+1}, \dots, v_{q+r+s}\}$ is a cycle in the acyclic graph \mathcal{G} .

– $u_k \prec_{\mathcal{G}} v \prec_{\mathcal{G}} u_{k+1}$ for some $k \in \llbracket 1, n-1 \rrbracket$. In this case, there exist two $s-t$ paths $\lambda^1 = \{v_1, \dots, v_q, u_k, v_{q+1}, \dots, v_{q+r}, v, v_{q+r+1}, \dots, v_{q+r+s}\} \in \Lambda$ and $\lambda^2 = \{w_1, \dots, w_n, v, w_{n+1}, \dots, w_{n+m}, u_{k+1}, w_{n+m+1}, \dots, w_{n+m+p}\} \in \Lambda$. One can verify that $\{v_{q+1}, \dots, v_{q+r}, v, w_{n+1}, \dots, w_{n+m}\}$ is a cycle in \mathcal{G} . This contradicts \mathcal{G} being acyclic.

Thus, $\lambda = C$ and $\Lambda \subseteq \mathcal{C}$. In conclusion, $\mathcal{C} = \Lambda$. \square

Figure A.3. (Color online) Proof of transitivity of \preceq_G : if $u \preceq_G v$, and $v \preceq_G w$, then one can construct an $s - t$ path λ^2 (in thick line) that traverses u and w in this order.



Appendix B. Illustration of Algorithm 1

Consider the poset P represented by the Hasse diagram given in Figure B.1.

In this poset P , the set of maximal chains is given by $\mathcal{C} = \{\{1, 3, 4\}, \{2, 3, 5\}, \{1, 3, 5\}, \{2, 3, 4\}\}$. We assume that the values assigned to each maximal chain are $\pi_{134} = \pi_{135} = 0.8$ and $\pi_{234} = \pi_{235} = 0.6$, and the values assigned to each element are $\rho_1 = 0.4, \rho_2 = 0.3, \rho_3 = 0.5, \rho_4 = 0.5, \rho_5 = 0.7$.

First, we can see that for all $C \in \mathcal{C}, \sum_{x \in C} \rho_x \geq \pi_C$ and $\pi_{134} + \pi_{235} = \pi_{135} + \pi_{234}$. Therefore, Conditions (2) and (3) are satisfied, and we can run Algorithm 1 to optimally solve (Q) (and construct a feasible solution of (D)). Figure B.2(a) (respectively, Figure B.2(b)), illustrates each iteration of the algorithm using the poset P (respectively, the posets P^k , for $k \in \llbracket 1, n^* \rrbracket$).

- $k = 1$: $X^1 = X = \llbracket 1, 5 \rrbracket, C^1 = \mathcal{C}, \rho_x^1 = \rho_x$ for all $x \in X$. Note that $\delta_{134} = 0.6, \delta_{235} = 0.9, \delta_{135} = 0.8,$ and $\delta_{234} = 0.7$. Since for all $C \in \mathcal{C}, \delta_C^1 = \delta_C > 0$, then $\bar{C}^1 = \emptyset$, and $\hat{C}^1 = \mathcal{C}$. Therefore, each pair of elements in $P^1 = (X^1, \preceq_{\bar{C}^1})$ is incomparable and $S^1 = \{1, 2, 3, 4, 5\}$. Then one can check that $\min_{x \in S^1} \rho_x^1 = 0.3$ and $\min_{\{C \in \mathcal{C}^1 \mid |S^1 \cap C| \geq 2\}} \frac{\delta_C^1}{|S^1 \cap C| - 1} = 0.3$. Therefore, $\sigma_{S^1} = w^1 = 0.3 = \rho_2^1 = \frac{\delta_{134}^1}{|S^1 \cap \{1, 3, 4\}| - 1}$.

Next, the values are updated as follows: $\rho_1^2 = 0.1, \rho_2^2 = 0, \rho_3^2 = 0.2, \rho_4^2 = 0.2, \rho_5^2 = 0.4,$ and $\delta_{134}^2 = 0, \delta_{235}^2 = 0.3, \delta_{135}^2 = 0.2, \delta_{234}^2 = 0.1$. Since each maximal chain's minimal element is in S^1 , then $\mathcal{C}^2 = \mathcal{C}$. We conclude the first iteration of the algorithm by letting $X^2 = \{1, 3, 4, 5\}, \bar{C}^2 = \{\{1, 3, 4\}\},$ and $\hat{C}^2 = \{\{2, 3, 5\}, \{1, 3, 5\}, \{2, 3, 4\}\}$.

- $k = 2$: The set of minimal elements of the new poset $P^2 = (X^2, \preceq_{\bar{C}^2})$ is given by $S^2 = \{1, 5\}$. Furthermore, $\min_{x \in S^2} \rho_x^2 = 0.1$ and $\min_{\{C \in \mathcal{C}^2 \mid |S^2 \cap C| \geq 2\}} \frac{\delta_C^2}{|S^2 \cap C| - 1} = 0.2$, which imply that $\sigma_{S^2} = w^2 = 0.1 = \rho_1^2$. Then, the values are updated as follows: $\rho_1^3 = 0, \rho_2^3 = 0, \rho_3^3 = 0.2, \rho_4^3 = 0.2, \rho_5^3 = 0.3,$ and $\delta_{134}^3 = 0, \delta_{235}^3 = 0.3, \delta_{135}^3 = 0.1, \delta_{234}^3 = 0.1$.

Now, one can see that the minimal element of $\{2, 3, 5\} \cap X^2$ and $\{2, 3, 4\} \cap X^2$ in P is $3 \notin S^2$. Therefore, $\mathcal{C}^3 = \{\{1, 3, 4\}, \{1, 3, 5\}\}, X^3 = \{3, 4, 5\}, \bar{C}^3 = \{\{1, 3, 4\}\},$ and $\hat{C}^3 = \{\{1, 3, 5\}\}$.

- $k = 3$: The set of minimal elements of $P^3 = (X^3, \preceq_{\bar{C}^3})$ is given by $S^3 = \{3, 5\}$. Since $\min_{x \in S^3} \rho_x^3 = 0.2$, and $\min_{\{C \in \mathcal{C}^3 \mid |S^3 \cap C| \geq 2\}} \frac{\delta_C^3}{|S^3 \cap C| - 1} = 0.1$, then $\sigma_{S^3} = w^3 = 0.1 = \frac{\delta_{135}^3}{|S^3 \cap \{1, 3, 5\}| - 1}$. The values are updated as follows: $\rho_1^4 = 0, \rho_2^4 = 0, \rho_3^4 = 0.1, \rho_4^4 = 0.2, \rho_5^4 = 0.2,$ and $\delta_{134}^4 = 0, \delta_{235}^4 = 0.2, \delta_{135}^4 = 0, \delta_{234}^4 = 0.1$. Then, $X^4 = \{3, 4, 5\}, C^4 = \mathcal{C}^3, \bar{C}^4 = \{\{1, 3, 4\}, \{1, 3, 5\}\},$ and $\hat{C}^4 = \emptyset$.

- $k = 4$: The set of minimal elements of $P^4 = (X^4, \preceq_{\bar{C}^4})$ is $S^4 = \{3\}$. Then, $\sigma_{S^4} = w^4 = \min_{x \in S^4} \rho_x^4 = \rho_3^4 = 0.1$, and the new values are $\rho_1^5 = 0, \rho_2^5 = 0, \rho_3^5 = 0, \rho_4^5 = 0.2, \rho_5^5 = 0.2,$ and $\delta_C^5 = \delta_C^4$ for all $C \in \mathcal{C}$. The new sets are $X^5 = \{4, 5\}, C^5 = C^4, \bar{C}^5 = \{\{1, 3, 4\}, \{1, 3, 5\}\},$ and $\hat{C}^5 = \emptyset$.

- $k = 5$: The set of minimal elements of $P^5 = (X^5, \preceq_{\bar{C}^5})$ is given by $S^5 = \{4, 5\}$, and the weight associated with it is $\sigma_{S^5} = w^5 = \rho_4^5 = \rho_5^5 = 0.2$. Then, $\rho_x^6 = 0$ for all $x \in X$, and $\delta_C^6 = \delta_C^5$ for all $C \in \mathcal{C}$.

Since $X^6 = \emptyset$, the algorithm terminates and outputs σ , which satisfies Constraints (5) and (6). The total weight utilized is $\sum_{S \in \mathcal{P}} \sigma_S = 0.8 = \max\{\max\{\rho_x, x \in X\}, \max\{\pi_C, C \in \mathcal{C}\}\}$. Therefore, from Theorem 2, σ is an optimal solution of (Q). Since $0.8 \leq 1$, then $\hat{\sigma} \in \mathbb{R}_{\geq 0}^{\mathcal{P}}$ given by $\hat{\sigma}_S = \sigma_S$ for every $S \in \mathcal{P} \setminus \emptyset$, and $\hat{\sigma}_\emptyset = 0.2$ is a feasible solution of (D).

Figure B.1. Hasse diagram of a poset P .

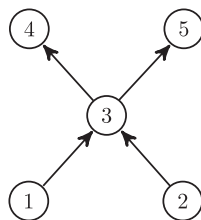
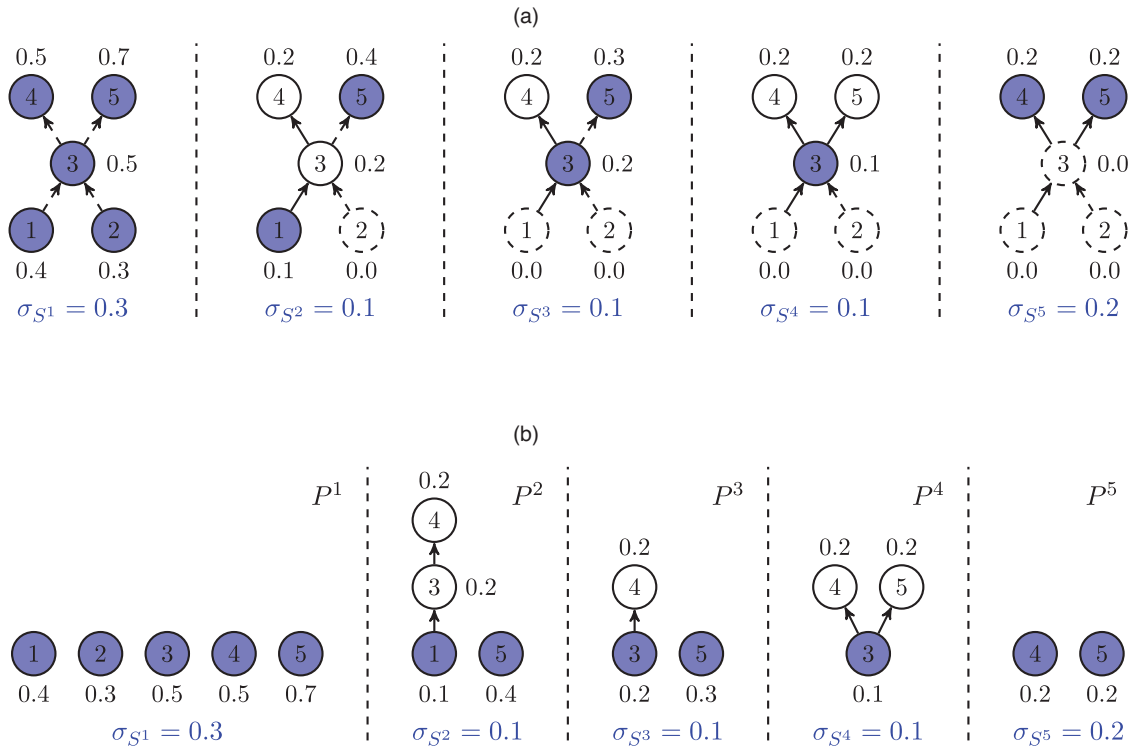


Figure B.2. (Color online) Illustration of Algorithm 1 for the poset P given in Figure B.1. (a) Poset P at the beginning of each iteration of the algorithm. The solid nodes are in X^k , the dashed nodes are in $X \setminus X^k$, and the filled nodes are in S^k . An edge is solid if there exists a maximal chain in \mathcal{C}^k that contains both end nodes of the edge. The values ρ_x^k are given next to each element. (b) P^k , for $k \in \llbracket 1, 5 \rrbracket$. The values ρ_x^k are given next to each element. S^k is given by the filled nodes.



Appendix C. Minimum-Cost Circulation Problem

Primal and dual linear formulations of (\mathcal{M}) of polynomial size are given as follows:

$$\begin{aligned}
 (\mathcal{M}'_P) : \quad & \text{maximize} && \sum_{\{i \in \mathcal{V} \mid (i,t) \in \mathcal{E}\}} f_{it} - \sum_{(i,j) \in \mathcal{E}} \frac{b_{ij}}{p_1} f_{ij} \\
 & \text{subject to} && \sum_{\{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}} f_{ji} = \sum_{\{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}} f_{ij}, \quad \forall i \in \mathcal{V} \setminus \{s,t\} \\
 & && 0 \leq f_{ij} \leq c_{ij}, \quad \forall (i,j) \in \mathcal{E} \\
 & && 0 \leq f_{ij} \leq \frac{d_{ij}}{p_2}, \quad \forall (i,j) \in \mathcal{E}.
 \end{aligned}$$

$$\begin{aligned}
 (\mathcal{M}'_D) : \quad & \text{minimize} && \sum_{(i,j) \in \mathcal{E}} c_{ij} \rho_{ij} + \frac{d_{ij}}{p_2} \mu_{ij} \\
 & \text{subject to} && y_i - y_j + \rho_{ij} + \mu_{ij} \geq -\frac{b_{ij}}{p_1}, \quad \forall (i,j) \in \mathcal{E} \mid i \neq s \text{ and } j \neq t \\
 & && -y_j + \rho_{sj} + \mu_{sj} \geq -\frac{b_{sj}}{p_1}, \quad \forall j \in \mathcal{V} \mid (s,j) \in \mathcal{E} \\
 & && y_i + \rho_{it} + \mu_{it} \geq 1 - \frac{b_{it}}{p_1}, \quad \forall i \in \mathcal{V} \mid (i,t) \in \mathcal{E} \\
 & && \rho_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{E} \\
 & && \mu_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{E}.
 \end{aligned}$$

Let $z^*_{(\mathcal{M}'_P)}$ denote the optimal value of (\mathcal{M}'_P) and (\mathcal{M}'_D) . We show the following result:

Lemma C.1. Any $s-t$ path decomposition of any optimal solution f' of (\mathcal{M}'_P) is an optimal solution of (\mathcal{M}_P) . Furthermore, given any optimal solution (ρ', μ', y') of (\mathcal{M}'_D) , (ρ', μ') is an optimal solution of (\mathcal{M}_D) .

Proof of Lemma C.1. Let $f^* \in \mathbb{R}_{\geq 0}^{\Lambda}$ be an optimal solution of (\mathcal{M}_P) . Then, $f' \in \mathbb{R}_{\geq 0}^{\mathcal{E}}$ defined by $f'_{ij} = \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f^*_\lambda$ is a feasible solution of (\mathcal{M}'_P) . Therefore, $z^*_{(\mathcal{M}')} \geq z^*_{(\mathcal{M})}$. Now, let $f' \in \mathbb{R}_{\geq 0}^{\mathcal{E}}$ be an optimal solution of (\mathcal{M}'_P) . From the flow decomposition theorem, there exists a vector $f^* \in \mathbb{R}_{\geq 0}^{\Lambda}$ such that for all $(i,j) \in \mathcal{E}$, $f'_{ij} = \sum_{\{\lambda \in \Lambda \mid (i,j) \in \lambda\}} f^*_\lambda$. Since f^* is a feasible solution of (\mathcal{M}_P) , then $z^*_{(\mathcal{M})} \geq z^*_{(\mathcal{M}')}.$ In conclusion, $z^*_{(\mathcal{M})} = z^*_{(\mathcal{M}')}.$ and an optimal solution of (\mathcal{M}_P) can be obtained by decomposing an optimal solution of (\mathcal{M}'_P) into $s-t$ paths.

Now, consider an optimal solution (ρ', μ', y') of (\mathcal{M}'_D) . Then, one can verify that for every $s-t$ path $\lambda \in \Lambda$, $\sum_{(i,j) \in \lambda} (\rho'_{ij} + \mu'_{ij}) \geq 1 - \sum_{(i,j) \in \lambda} b_{ij}/p_1 = \pi_\lambda^0$ (the y' cancel in a telescopic manner along each $s-t$ path). Therefore, (ρ', μ') is a feasible solution of (\mathcal{M}_D) . Because $z^*_{(\mathcal{M}')} = z^*_{(\mathcal{M})}$, we conclude that (ρ', μ') is an optimal solution of (\mathcal{M}_D) . \square

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