# Risk-based Manpower Planning
## A Tractable Multi-period Model

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The manpower planning problem of hiring and promoting has been the perennial difficulty of HR management. We propose a risk-based approach – finding a course of action that provides guarantees against the risk of running short of organizational targets, such as productivity, budget, headcount and managerial span of control. As such, this approach leads to an optimization model that minimizes a risk parameter, inspired by Aumann and Serrano (2008)'s riskiness index. Additionally, our model departs from the literature by considering employees' time-in-grade, which is known to affect resignations, as a decision variable. In our formulation, decisions and the uncertainty are related. To solve the model, we introduce the technique of *pipeline invariance*, which yields an exact re-formulation that may be tractably solved. Computational performance of the model is studied by running simulations on a real dataset of employees performing the same job function in the Singapore Civil Service. Using our model, we are able to illustrate insights into HR, such as the consequences of a lack of organizational renewal. Our model is also likely the first numerical illustration that lends weight to a time-based progression policy common to bureaucracies. We believe that this technique of pipeline invariance could help solve a wider range of multi-period optimization problems.

*Key words*: manpower planning; robust optimization; multi-period optimization; public sector; riskiness index

## 1. Introduction

Of late, the Human Resource (HR) function is gaining prominence. This is driven by the growing practice of Strategic HR Management (SHRM) (Ulrich and Dulebohn 2015, Buyens and De Vos 2001), where human capital is structured to achieve transformational goals of the organization. In addition, data analytics is enabling HR practitioners to consider costs and returns of investment (ROI) of human capital as business decisions. This is evidenced by how HR analytics has been introduced in almost every aspect of HR Management (Davenport et al. 2010). Examples include attrition and flight risk, talent and pipeline management, recruitment analytics and employee value

proposition, under-performance risks, remuneration and benefits, real-time employee engagement and sentiment analysis, learning and gamification in the workplace, team performance and social networks, to name only a few.

Nonetheless, HR analytics continues to struggle to draw the link between human capital and organizational outcomes (Marler and Boudreau 2017, Angrave et al. 2016). Herein lies the irony – despite possessing more information than ever, HR has found itself increasingly in a state of *decision paralysis* (Kapoor and Kabra 2014). While data abounds, there remains a lack of an operational frame integrating these data and secondary analyses into trade-offs and risks at the organizational level. As such, few practitioners have managed to extend the current advances in predictive analytics into the realm of prescriptive analytics.

In this paper, we hope to make some preliminary steps towards this overarching goal. In particular we would like to concentrate on the topic of manpower planning – how should a business unit hire, promote, and design its operational structure in order to achieve a targeted productivity level, while constrained by budget, availability of manpower and managerial span of control? This is not simple; the trade-offs between different HR decisions may not be at first glance apparent. For example, the optimal staffing level across different competency bands could depend on both the productivity targets that the organization aspires to meet and the expectation of employees on promotion and remuneration.

We approach this problem from a risk perspective, to be detailed later. The logic comes from the fact that different HR interventions cannot be assessed separately as it is their combination that affects employees and their behaviors. For example, an employee's career management can have downstream effects on their flight risk, under-performance risk, engagement levels, etc.As such, it may be possible to perceive HR management as a basket of interventions articulating into eventual outcomes of individuals and organizational units, the risk of which we seek to minimize (Paul and Mitlacher 2008).

**HR in Public Services**

We focus on manpower planning within public sector organizations (PSOs). In PSOs, HR decisions may be constrained by many different and specific considerations from the private sector. This is because of differences in risk attitudes (Nutt 2006, Rodrigues and Hickson 1995) – given the risk of government failure and the fact of being embedded in the political landscape, PSOs are exposed to greater downside risks than profit-seeking firms. This can limit HR in its perceived action space. For example, many PSOs may find it difficult to voluntarily release their employees, either for fear of political backlash or where such actions, as is the case in many countries, are constrained by

the action of unions. We shall, in a later analysis, examine how this lack of organizational renewal affects the risks of operations.

Another key distinction is the difficulty of measuring productivity and ROI of public service employees. Public service outcomes can be ambiguous. Managers may at times grant heavier consideration to the perceived outcomes of policies (to the polity) instead of their actual outcomes (Heinrich and Marschke 2010). The long length of policy maturity too can make it difficult to assess the impacts of policies, and to link them to decisions or actions taken by specific individuals. These challenges in turn shape performance management and incentive structures, for example, in public sector innovation (Walker et al. 2011). As such, it can be difficult to measure performances and wages in the same units, or to describe their trade-offs.

These challenges describe an operating context wherein the objective functions are multiple, competing and ambiguous. Instead, we posit that PSOs may best be characterized as trying to balance a set of alternatives, so as to avoid the emergence of adverse circumstances. This philosophy is fundamentally risk-based in nature and this motivates the paradigm that we shall adopt.

In response, many PSOs employ a *time-based progression policy*. This is a dogma where employees are promoted only after spending a minimum amount of time at their present grade. Apart from skirting the issue of having to directly measure the productivity of employees, it also embodies a certain operational logic – an employee must spend a minimum amount of time learning the ropes, before maturing as an active contributor. Conversely, remaining too long in the job may lead to boredom, apathy, entrenchment and disengagement. As such, there is an optimal time to promote employees. The goal of our paper is to study this.

**Optimization in HR Management**

The manpower planning problem is not new. Over the last half a century or so, there have been various approaches, such as a simulation-based or systems dynamics approach (to raise a few examples: Park et al. 2008, Chung et al. 2010), an econometric approach (*e.g.* Roos et al. 1999, Sing et al. 2012), and finally, a mathematical programming approach, which is the focus of this paper.

The most popular approach has been from the perspective of a Markov model. Bartholomew et al. (1991) provides a broad overview. Various improvements over the years have incorporated learning effects (*e.g.* Gans and Zhou 2002), inter-departmental flows (Song and Huang 2008), and staff scheduling (such as Abernathy et al. 1973) just to name a few extensions. The primary goal of the Markov model is to set up the transition probabilities through the hierarchy and determine the two central questions of attainability (Is it possible to transit from one organization of work to another?) and sustainability (What is the minimum cost to do so?). As explained in Gurrey and

De Feyter (2012), attainability is not always guaranteed. As such, additional conditions (such as the proportionality assumption in Nilakantan and Raghavendra 2004) and approximate measures (such as fuzzy sets as in Dimitriou et al. 2013) have been introduced.

At the broader level, some researchers have moved away from the Markov paradigm and approached the problem via dynamic programming (as in Mehlmann 1980, Flynn 1981, Rao 1990). In order to balance between competing organizational outcomes, some have adopted a goal programming paradigm (Price and Piskor 1972, Georgiou and Tsantas 2002). In more modern literature, researchers have applied stochastic programming techniques supported by linearisations and Bender's decomposition as in Zhu and Sherali (2009), in order to tackle the computational difficulties.

Nonetheless, these methods suffer from the curse of dimensionality, and become rapidly unscalable with the number of input variables. For example, in Zhu and Sherali's case, the stochastic model only solved three out of ten times in computational tests. In the age of data analytics, taking as input individual-level machine learning predictions of flight risk and performance risk would very likely exceed the computational limits of these models. Moreover, it is not immediately apparent how these models can be extended to be robust to uncertainty, especially resignations, which are known to fluctuate wildly.

Most critically, time spent by an employee in a grade (time-in-grade, for short) is often ignored, though it is known to be a major determinant of employee behavior, such as resignations. Incorporating them however poses challenges – the uncertainty at each time period will depend on decisions made in the previous period. Hence, techniques to deal with it are few and far in between.

One of the earliest attempts was by Bres et al. (1980), which presented a linear goal programming model that decided on the number of promotions where time-in-organization was a factor. Subsequently, Kalamatianou (1987) retained the Markov framework, by cutting up the population of employees into those yet-to-be and those ready-to-be-promoted, and estimating the transition probabilities based on the age distribution. Nonetheless, this did not directly address the interdependence of decision and uncertainty and was a workaround. Finally, Nilakantan and Raghavendra (2008) attempted a Markov model based on both time-in-grade and time-in-organization, but only under strict assumptions. Unfortunately, they also stopped short of attainability.

This lack of computational tractability and modeling flexibility limit the ability of HR practitioners to implement a recruitment and progression strategy based on time-in-grade. At its base, there isn't even conclusive numerical evidence in support of time-based progression, which is practiced across many bureaucracies. We aim to fill this gap in this paper.

To address these challenges, we propose to consider approaches based on robust optimization. Over the last decade and a half, modern robust optimization techniques have successfully been

introduced in many disciplines, from logistics and supply chain management to healthcare operations and to transportation, etc. Its popularity comes from its ability to factor in the uncertainty while preserving the complexity class of the original deterministic problem. This, in practice, leads to scalable solutions (Ben-Tal et al. 2004, Bertsimas and Sim 2004). The flexibility of robust optimization to prescribe different forms of uncertainty, such as in the distributional sense (Delage and Ye 2010, Wiesemann et al. 2014) and fully data-driven sense (*e.g.* using the Wasserstein metric in Mohajerin Esfahani and Kuhn 2017) is another contributing factor.

In manpower planning, robust optimization has traditionally been applied to staffing and scheduling problems (for example, Burke et al. 2004, Lusby et al. 2012). However, to the best of our knowledge, we haven't seen any literature on its application to longer term manpower planning.

## Contributions of this Paper

First, we propose a novel risk-based multi-period optimization framework where the uncertainty and decision space have a specific inter-dependent structure, termed 'pipeline invariance'. This improves on the pre-existing literature:

1. The model circumvents the prescription of trade-offs between outcomes and policies by instead requiring the articulation of risk attitudes for each macro-objective.
2. It provides guarantees against constraint violation.
3. It can be approximated to high order by a second-order conic program (SOCP), hence solved efficiently despite the inter-dependence of uncertainty and decisions.
4. It may be reasonably extended to incorporate data at the individualized level, which can take as input the results from various predictive analytics models.

Second, we claim that our model provides a novel and useful application in the domain of HR. We shall illustrate insights drawn from our approach, in particular, giving quantitative substantiation for a time-based progression model and the ramifications of a lack of organizational renewal. The model has since been applied to study progression in a job function in the Singapore Civil Service. Aside from HR, we also believe that our model can be applied more broadly in other multi-period risk-centric contexts.

## Notation

Given $N \in \mathbb{N}$, let $[N]$ represent $\{1, \ldots, N\}$ and denote $[N]_0 := \{0\} \cup [N]$. Let $\mathbb{R}_0^+$ be the non-negative reals. We shall use bold-faced characters such as $\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{A} \in \mathbb{R}^{M \times N}$ to represent vectors and matrices. We use $x_i$ to denote the $i$-th element of vector $\boldsymbol{x}$. The tilde sign shall denote an uncertain or random parameter such as $\tilde{z}$ without explicitly stating its probability distribution. We use the convention, $\log 0 = \max \emptyset = -\infty$ and $\min \emptyset = \infty$.

We shall use $\mathbb{E}[\cdot]$ to represent the expectation over the uncertainty across all time periods, unless otherwise stated. Specifically, we shall use $\mathbb{E}_t[\cdot]$ to abbreviate the conditional expectation over time $t$, given materializations of the uncertainty up to time $t-1$. Consequently, after evaluating $\mathbb{E}_t[\cdot]$ we also denote $\mathbb{E}_{\leq t-1}[\cdot]$ as the expectation over all uncertainty up to time $t-1$.

## 2. A Model for Manpower Planning

Traditionally, the manpower planning problem is set up over a finite time horizon $t \in [T]_0$, where $t = 0$ is the present data and $t = T$ is the last time period to be considered. Often, the objective is to attain a known staffing level. Let the *stock* $(\tilde{\boldsymbol{s}}^t)_l^\tau := \tilde{s}_l^{t,\tau}$ denote the number of employees at time $t$ for all the times up to the last planning time $T$, having spent $\tau \in [M]_0$ years at grade $l \in [L]$. Each is paid wage $w_l^\tau$ to generate a return of productivity $r_l^\tau$. Let the decision variables $h_l^t$ denote the number of employees hired at time $t$ into grade $l$ and $p_l^{t,\tau}$ officers to be promoted from grade $l$ to $l+1$ for $l \in [L-1]$.

The *organizational structure* is the hierarchy of grades. Similar to existing literature, we categorize individual contributors into skills strata $l \in \mathcal{W} := [L_w]$, where $L_w$ is the highest skills stratum. These contributors are supervised by managers, limited by the maximum number of employees they can manage, called the span of control $c_l^\tau$. In our model, managers occupy the higher grades $l \in \mathcal{M} := \{L_w + 1, \ldots, L\}$, where $L$ is the highest grade in the hierarchy. Promotion is thus the movement of employees between adjacent strata. For simplicity, assume that promotion only occurs between adjacent grades and ignore complications, such as transfers across departments.

Employees may be lost through *attrition*. In the literature, attrition is often understood as a rate – an annual proportion of stock $\tilde{\boldsymbol{s}}$. Instead we hope to model attrition as a random variable depending on the decision variables, so as to capture the inter-dependence – employees who were not promoted have a different chance of leaving compared to those who were.

More specifically, given state space $\mathcal{X} \subseteq \mathbb{R}_0^+$ containing stock $\tilde{s}_l^{t,\tau}$, define attrition as the collection of random variables mapping from $\mathcal{X}$ to itself, $A_l^\tau(\omega; x) : \Omega \times \mathcal{X} \to \mathcal{X}$ measurable under some probability space $(\Omega, \Sigma, \mathbb{P})$ for all $x \in \mathcal{X}$. Hence, $x$ is the stock prior to the onset of attrition. For brevity, we use the notation $\tilde{s}_l^{t,\tau} = \tilde{Z}_l^\tau(x)$ to denote this dynamics $\tilde{s}_l^{t,\tau} \,|\, x \sim A_l^\tau(\cdot, x)$. In our case, we consider the binomial random variable $\mathrm{Bin}(x, q)$, where the number of trials $x$ represents the stock and the success probability $q := q_l^\tau$ represents the chance an employee stays within an organization till the next year (also called 'retention').

Other than capturing the inter-dependence of uncertainty and decisions, this definition also affords us further benefits. Distributions otherwise difficult to represent if attrition were expressed as a rate, can now be described. For example, the $O(x)$ behavior of the variance in the binomial distribution, $\mathrm{Bin}(x, q)$, can now be captured. Moreover, the model can be extended to the case

where individualized data is supplied. Here, $q$ can be the predicted flight risk of individual employees under separately constructed predictive models (*e.g.* a random forest model).

With this, we define the dynamics which depends on time-in-grade $\tau \in [M]_0$:

$$
\begin{align}
\tilde{s}_{l+1}^{t,0} &= h_{l+1}^t + \sum_{\tau'} p_l^{t-1,\tau'} && \forall t \in [T], \forall l \in [L-1] \\
\tilde{s}_l^{t,\tau} &= \tilde{Z}_l^\tau(\tilde{s}_l^{t-1,\tau-1} - p_l^{t-1,\tau-1}) && \forall t \in [T], \forall l \in [L-1], \forall \tau \in [M] \\
\tilde{s}_L^{t,\tau} &= \tilde{Z}_L^\tau(\tilde{s}_L^{t-1,\tau-1}) && \forall t \in [T], \forall \tau \in [M]
\end{align}
\tag{1}
$$

The first equation states that the employees new to grade $l+1$ at time $t$ are the totality of new hires and those promoted. The next two equations describe the erosion of the stock by promotions and attrition at each time period. The model is also accompanied by the following constraints:

1. *Headcount constraint*: $\sum_{l,\tau} \tilde{s}_l^{t,\tau} \leq H_t$, $\forall t \in [T]$.
2. *Budget constraint*: $\sum_{l,\tau} \tilde{s}_l^{t,\tau} w_l^\tau \leq B_t$, $\forall t \in [T]$.
3. *Productivity constraint*: $\sum_{l,\tau} \tilde{s}_l^{t,\tau} r_l^\tau \geq P_t$, $\forall t \in [T]$.
4. *Span of control constraint*: For each $l \in \mathcal{M}$, let $\mathcal{W}_l \subseteq [l-1]$ be the employee grades supervised by the manager. Then $\sum_\tau \tilde{s}_l^{t,\tau} c_l^\tau \geq \sum_{\substack{\lambda \in \mathcal{W}_l \\ \tau}} \tilde{s}_\lambda^{t,\tau}$, $\forall l \in \mathcal{M}$, $\forall t \in [T]$. We can simplify this by letting

$$
b_{l,\lambda}^\tau = \begin{cases} c_l^\tau & \text{if } \lambda = l \\ -1 & \text{if } \lambda \in \mathcal{W}_l \\ 0 & \text{otherwise} \end{cases}
$$

then the constraint may simply be written as $\sum_{\lambda,\tau} \tilde{s}_\lambda^{t,\tau} b_{l,\lambda}^\tau \geq 0$, $\forall t \in [T], \forall l \in \mathcal{M}$.

5. Trivial constraints $h_l^t \geq 0$ and $0 \leq p_l^{t,\tau} \leq \tilde{s}_l^{t,\tau}$ for $\forall t \in [T], \forall l \in [L], \forall \tau \in [M]$.

## A Risk-based Model

In the literature, one might minimize the costs of maintaining a workforce, maximize the total productivity of employees, or deal with these multiple objectives in the goal programming sense (for example in Price and Piskor 1972). However, it could be difficult to prescribe the trade-offs between costs and productivity (*e.g.* for a maintenance crew), say in goal programming.

It may also be more appropriate within some business contexts neither to maximise output nor minimise operating costs, but to run the least risk of disruption, such as a service centre. This approach is especially relevant for public sector organizations. Without a clear objective, we instead pursue an optimization model which minimizes this risk. It sounds tempting to minimize the joint probability of constraint violation, but this leads to a chance-constrained program, which, often and in particular in this case, has intractable formulations. In fact, our goal doesn't necessitate the minimization of the chance of constraint violation *per se*. Instead, we simply desire a policy that

*does not fare too poorly*, in other words, a course of action with some guarantees over the risks of violation.

Aumann and Serrano (2008)'s riskiness index has this functionality. Let $\mathcal{Z}$ be the set of all random variables on our probability space $(\Omega, \Sigma, \mathbb{P})$. Define the Aumann and Serrano (2008)'s riskiness index as the functional $\mu : \mathcal{Z} \to \mathbb{R}_0^+ \cup \{\infty\}$:

$$\mu(\tilde{z}) = \inf \left\{ k > 0 : \mathbb{E} \exp(\tilde{z}/k) \leq \exp(0) = 1 \right\}. \tag{2}$$

Here, $\tilde{z}$ represents the size of the violation – a positive number constitutes a violation and vice versa. The exponential disutility penalizes ever larger violations. The riskiness index gives the risk level $k$, where the expected disutility is equal to a state where there are no violations. $\mu$ may also be expressed as:

$$\mu(\tilde{z}) = \inf \left\{ k > 0 : C_{k,1}(\tilde{z}) \leq 0 \right\}, \tag{3}$$

in terms of the certainty equivalence

$$C_{k,\theta}(x) = k \log \left( \mathbb{E} \left[ \exp \left( x/k\theta \right) \right] \right).$$

The riskiness index obeys the following properties:

1. Satisficing: $\mu(\tilde{z}) = 0$ if and only if $\mathbb{P}[\tilde{z} \leq 0] = 1$.
2. Infeasibility: If $\mathbb{E}[\tilde{z}] > 0$, then $\mu(\tilde{z}) = \infty$.
3. Convexity: $\mu$ is convex in $\tilde{z}$.
4. Violation Guarantees: For $\mu(\tilde{z}) > 0$, $\mathbb{P}[\tilde{z} > \phi] \leq \exp(-\phi/\mu(\tilde{z}))$.

The first property states that there is no risk if there is no chance of violation. The second dictates that if violations are always expected, then the risk is always infinite. The third requires convexity and the fourth is our desired guarantee against constraint violation, which is the consequence of Markov's inequality. Hence $\mu(\tilde{z})$ captures the notion of risk – the lower $\mu(\tilde{z})$ is, the sharper the guarantee against ever larger violations $\phi$ of the constraint.

Aumann and Serrano (2008)'s riskiness index fits well to our setting – we seek a risk level $k$, such that the riskiness index of every constraint is bounded by at most $k$. As a consequence, we obtain guarantees against the chance of constraint violations for each constraints. This risk level $k$ hence is the objective that we seek to minimize.

Monotonicity of $C_{k,\theta}$ in $k$ allows us to relax each linear constraint $\tilde{x} \leq G$ to $C_{k,\theta}(\tilde{x} - G) \leq 0$. As $C_{k,\theta}$ is a perspective map in $k$, it is jointly convex in $k$ and $x$. As such, this relaxation preserves convexity. The constraints can now be violated, but the level of violation is controlled for $\phi > 0$,

$$\mathbb{P}[\tilde{x} - G \geq \phi] \leq \exp(-\phi/k\theta).$$

Note that $\theta$ acts as a tuning parameter, which behaves like $k$. As such, $\theta$ dictates the tightness of this exponential envelope – a user may prescribe the degree of tightness based on the acceptable level of risk for that constraint. Across constraints, $\theta$ calibrates the relative risk aversion of violating each constraint, for example, say to emphasise that the budget constraint is more critical than the headcount, or across time, such as a stronger aversion to earlier time violation than in the future as with discounting. For simplicity, we shall assume $\theta = 1$ in all subsequent discussions and is dropped from further notation, until later in an analysis where we have to calibrate the tightness of the hiring constraint (in page 24). The alternative to prescribing $\theta$ is to perform lexicographic minimization. More shall be said about that later.

Before moving on, we make a final adjustment to represent promotion as a rate of stock instead of an absolute figure. In particular, this changes the dynamics (1) to:

$$
\begin{aligned}
\tilde{s}_{l+1}^{t,0} &= h_{l+1}^t + \sum_{\tau} \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t-1,\tau-1} - d_l^{t,\tau}}{d_l^{t-1,\tau-1}} && \forall t \in [T], \forall l \in [L-1] \\
\tilde{s}_l^{t,\tau} &= \tilde{Z}_l^\tau \left( \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t,\tau}}{d_l^{t-1,\tau-1}} \right) && \forall t \in [T], \forall l \in [L-1], \forall \tau \in [M] \qquad (4)\\
\tilde{s}_L^{t,\tau} &= \tilde{Z}_L^\tau(\tilde{s}_L^{t-1,\tau-1}) && \forall t \in [T], \forall \tau \in [M]
\end{aligned}
$$

We also relax the requirement $h_{l+1}^t \geq 0$, replacing it with

$$
C_k \left( \sum_{\tau} \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t-1,\tau-1} - d_l^{t,\tau}}{d_l^{t-1,\tau-1}} - \tilde{s}_{l+1}^{t,0} \right) \leq 0. \qquad (5)
$$

This increases the flexibility of the model to 'fire' employees, while still remaining averse to releasing a large number of them. In this case, $s_l^{t,0}$ becomes a decision variable. This has the desirable side effect of reformulating $0 \leq p_l^{t,\tau} \leq \tilde{s}_l^{t,\tau}$ in terms of decision variables only $d_l^{t-1,\tau-1} \geq d_l^{t,\tau} \geq 0$.

We can now state our Risk-based Manpower Planning (RMP) model:

$$
\begin{aligned}
\min \quad & k \\
\text{s.t.} \quad & \tilde{s}_l^{t,\tau} = \tilde{Z}_l^\tau \left( \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t,\tau}}{d_l^{t-1,\tau-1}} \right) && \forall t \in [T], \forall l \in [L-1], \forall \tau \in [M] \\
& \tilde{s}_L^{t,\tau} = \tilde{Z}_L^\tau(\tilde{s}_L^{t-1,\tau-1}) && \forall t \in [T], \forall \tau \in [M] \\
& k \log \left( \mathbb{E}\left[ \exp \left( \left(\textstyle\sum_{l,\tau} \tilde{s}_l^{t,\tau} - H_t \right)/k \right) \right] \right) \leq 0 && \forall t \in [T] \\
& k \log \left( \mathbb{E}\left[ \exp \left( \left(\textstyle\sum_{l,\tau} \tilde{s}_l^{t,\tau} w_l^\tau - B_t \right)/k \right) \right] \right) \leq 0 && \forall t \in [T] \qquad (6)\\
& k \log \left( \mathbb{E}\left[ \exp \left( \left(P_t - \textstyle\sum_{l,\tau} \tilde{s}_l^{t,\tau} r_l^\tau \right)/k \right) \right] \right) \leq 0 && \forall t \in [T] \\
& k \log \left( \mathbb{E}\left[ \exp \left( -\textstyle\sum_{\lambda,\tau} \tilde{s}_\lambda^{t,\tau} b_{l,\lambda}^\tau /k \right) \right] \right) \leq 0 && \forall t \in [T], \forall l \in \mathcal{M}
\end{aligned}
$$

$$k \log \left( \mathbb{E}\left[ \exp\left( \left( \sum_\tau \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t-1,\tau-1} - d_l^{t,\tau}}{d_l^{t-1,\tau-1}} - s_{l+1}^{t,0} \right)\big/k \right) \right] \right) \le 0 \quad \forall t \in [T], \forall l \in [L]$$

$$s_l^{t,0} \ge 0,\ d_l^{t-1,\tau-1} \ge d_l^{t,\tau} \ge 0 \qquad\qquad\qquad \forall t \in [T], \forall l \in [L], \forall \tau \in [M]$$

**Pipeline Invariance**

This model turns out to be tractable. We first state a property:

**Property 1 (Pipeline Invariance)** *Let $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathbb{R}$, where $\mathcal{Y}$ is closed under multiplication. The $\mathcal{X}$-collection of random variables $A(\,\cdot\,;x): \Omega \times \mathcal{X} \to \mathcal{X}$, represented as $\tilde{Z}(x)$, satisfies* pipeline invariance *if there exists non-constant functions $\pi, \rho: \mathcal{Y} \to \mathcal{Y}$ such that*

$$\mathbb{E}\left[ \pi\big(y\tilde{Z}(x)\big) \right] = \pi\big(x \cdot \rho(y)\big), \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}, \tag{7}$$

*and $\rho(\cdot)$ is solely a function of $y$. We call $\pi$ and $\rho$ the* preservation function *and the* relay function *of $\tilde{Z}$ respectively. Moreover, if the family $\{\tilde{Z}_l\}$, each satisfying pipeline invariance, shares the same preservation function (but with possibly different relay functions), then we shall call this a* pipeline invariant family.

Pipeline invariance is effectively the preservation of the functional form of $\pi(\cdot)$ under the action of taking expectations. For the binomial random variable $\mathrm{Bin}(x, q)$ for fixed $q$, pipeline invariance is satisfied with preservation $\pi(x) = \exp(x)$ and relay $\rho(y) = \log(1 - q + qe^y)$. It turns out that Pipeline Invariance is satisfied by a larger class of distributions, such as the Poisson random variable $\mathrm{Pois}(x)$ with dependence on its rate $x$, or the Chi-squared distribution $\chi^2(df)$ with dependence on the degrees of freedom $df$. Moreover, their relay functions are convex over some domain.

Pipeline invariance lies at the heart of our core result:

THEOREM 1 **(Pipeline Reformulation).** *Let $\{\tilde{Z}_l^\tau\}$ be a pipeline invariant family with preservation function $\pi(\cdot) = \exp(\cdot)$. Suppose that $\forall l \in [L], \tau \in [M]$ the relay functions $\rho_l^\tau$ are convex on $\mathcal{Y} \supseteq \mathcal{X}$. If integrality of $\tilde{\boldsymbol{s}}^t$ is relaxed and if for each $t > 0$, $\tilde{s}_l^{t,\tau}$ are independent for all $l$ and $\tau$, then constraints of the form,*

$$k \log \left( \mathbb{E}\left[ \exp\left( \big(\sum_{l,\tau} \tilde{s}_l^{t,\tau} u_l^\tau - U_t\big)\big/k \right) \right] \right) \le 0 \tag{8}$$

*under $\tilde{s}_l^{t,\tau} = \tilde{Z}_l^\tau\left( \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t,\tau}}{d_l^{t-1,\tau-1}} \right)$, is equivalent to the reformulation*

$$\sum_l s_l^{t,0} u_l^0 + k \sum_{\substack{l \\ 1 < t' \le t}} \frac{s_l^{t'-1,0}}{d_l^{t'-1,0}} \xi_l^{t',1} + k \sum_{\substack{l \\ \tau \ge t}} \frac{s_l^{0,\tau-t}}{d_l^{0,\tau-t}} \xi_l^{1,\tau-t+1} \le U_t$$

$$d_l^{t,\tau} \rho_l^\tau\big({}^{u_l^\tau}\!/k\big) \le \xi_l^{t,\tau} \qquad\qquad\qquad \forall \tau \in [M]$$

$$d_l^{t',\tau} \rho_l^\tau\big(\xi_l^{t'+1,\tau+1}/d_l^{t',\tau}\big) \le \xi_l^{t',\tau} \qquad\qquad \forall t' \in [t-1], \tau \in [M-t+t']$$

$$\tag{9}$$

*Proof of Theorem 1.* We prove by induction for $j = 1, \ldots, t$ that the constraint (8) has the equivalent reformulation:

$$\sum_l s_l^{t,0} u_l^0 + \sum_{\substack{l \\ 1 < t' \le j}} \xi_l^{t-t'+2,1} \frac{\tilde{s}_l^{t-t'+1,0}}{d_l^{t-t'+1,0}} + k \log \left( \mathbb{E}_{\le t-j} \left[ \prod_{\substack{l \\ \tau > j-1}} \exp \left( \tilde{s}_l^{t-j,\tau-j} \frac{\xi_l^{t-j+1,\tau-j+1}}{d_l^{t-j,\tau-j}} \right) \right] \right) \le U_t$$

$$d_l^{t,\tau} \rho_l^\tau (u_l^\tau / k) \le \xi_l^{t,\tau} \qquad\qquad \forall \tau \in [M] \tag{10}$$

$$d_l^{t-t',\tau-t'} \rho_l^{\tau-t'} \left( \xi_l^{t-t'+1,\tau-t'+1} / d_l^{t-t',\tau-t'} \right) \le \xi_l^{t-t',\tau-t'} \qquad\qquad \forall t' \in [j-1], \forall \tau \in [M] - [t']$$

Indeed, when $j = t$, then (9) is recovered, with some manipulation of indices.

We first show the case when $j = 1$, *i.e.* that (8) has the reformulation

$$\sum_l s_l^{t,0} u_l^0 + k \log \left( \mathbb{E}_{\le t-1} \left[ \prod_{l,\tau > 0} \exp \left( \tilde{s}_l^{t-1,\tau-1} \frac{\xi_l^{t,\tau}}{d_l^{t-1,\tau-1}} \right) \right] \right) \le U_t$$

$$d_l^{t,\tau} \rho_l^\tau (\tfrac{u_l^\tau}{k}) \le \xi_l^{t,\tau} \qquad\qquad \forall \tau \in [M] \tag{11}$$

Now, evaluating (8) gives

$$k \log \left( \mathbb{E} \left[ \exp \left( (\sum_{l,\tau} \tilde{s}_l^{t,\tau} u_l^\tau)/k \right) \right] \right)$$

$$= k \log \left( \mathbb{E}_{\le t-1} \left[ \prod_{l,\tau} \mathbb{E}_t \left[ \exp \left( (\tilde{s}_l^{t,\tau} u_l^\tau)/k \right) \right] \right] \right) \tag{12}$$

$$= \sum_l s_l^{t,0} u_l^0 + k \log \left( \mathbb{E}_{\le t-1} \left[ \prod_{l,\tau > 0} \mathbb{E}_t \exp \left( \frac{u_l^\tau}{k} \tilde{Z}_l^\tau \left( \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t,\tau}}{d_l^{t-1,\tau-1}} \right) \right) \right] \right)$$

$$= \sum_l s_l^{t,0} u_l^0 + k \log \left( \mathbb{E}_{\le t-1} \left[ \prod_{l,\tau > 0} \exp \left( \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t,\tau}}{d_l^{t-1,\tau-1}} \rho_l^\tau \left( \frac{u_l^\tau}{k} \right) \right) \right] \right) \tag{13}$$

Here, we have used independence in (12) and then pipeline invariance in (13). Also notice that the first term consists simply of constants and decision variables, and hence exiting it from the expectation was valid. At this point, $\rho_l^\tau(u_l^\tau/k)$ is just simply a constant. Hence we can create the auxiliary variable $\xi_l^{t,\tau}$ and represent it as the epigraph $d_l^{t,\tau} \rho_l^\tau(\frac{u_l^\tau}{k}) \le \xi_l^{t,\tau}$. This proves the $j = 1$ case.

To prove the induction step going from $j$ to $j+1$, it suffices to prove that

$$k \log \left( \mathbb{E}_{\le t-j} \left[ \prod_{\substack{l \\ \tau > j-1}} \exp \left( \tilde{s}_l^{t-j,\tau-j} \frac{\xi_l^{t-j+1,\tau-j+1}}{d_l^{t-j,\tau-j}} \right) \right] \right) \le X \tag{14}$$

can be reformulated as

$$\sum_l \xi_l^{t-j+1,1} \frac{\tilde{s}_l^{t-j,0}}{d_l^{t-j,0}} + k \log \left( \mathbb{E}_{\le t-j-1} \left[ \prod_{l,\tau > j} \exp \left( \tilde{s}_l^{t-j-1,\tau-j-1} \frac{\xi_l^{t-j,\tau-j}}{d_l^{t-j-1,\tau-j-1}} \right) \right] \right) \le X$$

$$d_l^{t-j,\tau-j} \rho_l^{\tau-j} \left( \xi_l^{t-j+1,\tau-j+1} / d_l^{t-j,\tau-j} \right) \le \xi_l^{t-j,\tau-j} \qquad \forall \tau \in [M] - [j] \tag{15}$$

We evaluate the LHS of (14), again using independence and shifting decision variables out of the expectation in (16), and pipeline invariance in (17).

$$\sum_l \xi_l^{t-j+1,1} \frac{\tilde{s}_l^{t-j,0}}{d_l^{t-j,0}} + k \log \left( \mathbb{E}_{\le t-j-1} \left[ \prod_{l,\tau > j} \mathbb{E}_{t-j} \exp \left( \tilde{s}_l^{t-j,\tau-j} \frac{\xi_l^{t-j+1,\tau-j+1}}{d_l^{t-j,\tau-j}} \right) \right] \right) \tag{16}$$

$$= \sum_l \xi_l^{t-j+1,1} \frac{\tilde{s}_l^{t-j,0}}{d_l^{t-j,0}}$$

$$+ k \log \left( \mathbb{E}_{\leq t-j-1} \left[ \prod_{l,\tau>j} \mathbb{E}_{t-j} \exp \left( \frac{\xi_l^{t-j+1,\tau-j+1}}{d_l^{t-j,\tau-j}} \tilde{Z}_l^{\tau-j} \left( s_l^{t-j-1,\tau-j-1} \frac{d_l^{t-j,\tau-j}}{d_l^{t-j-1,\tau-j-1}} \right) \right) \right] \right)$$

$$= \sum_l \xi_l^{t-j+1,1} \frac{\tilde{s}_l^{t-j,0}}{d_l^{t-j,0}}$$

$$+ k \log \left( \mathbb{E}_{\leq t-j-1} \left[ \prod_{l,\tau>j} \exp \left( \tilde{s}_l^{t-j-1,\tau-j-1} \frac{d_l^{t-j,\tau-j}}{d_l^{t-j-1,\tau-j-1}} \rho_l^{\tau-j} \left( \xi_l^{t-j+1,\tau-j+1} / d_l^{t-j,\tau-j} \right) \right) \right] \right) \quad (17)$$

Now, notice that $d_l^{t-j,\tau-j} \rho_l^{\tau-j} \left( \xi_l^{t-j+1,\tau-j+1} / d_l^{t-j,\tau-j} \right)$ is jointly convex in both $d_l^{t-j,\tau-j}$ and $\xi_l^{t-j+1,\tau-j+1}$, as it is simply the perspective on $\rho_l^{\tau-j} (\xi_l^{t-j+1,\tau-j+1})$. Moreover, $\exp(\cdot)$, within which it is lying, is convex increasing. Hence, we may replace it with a new auxiliary variable $\xi_l^{t-j,\tau-j}$ and its epigraph to recover (15), as desired. This completes the proof for the induction. $\qquad \square$

Notice first that for fixed $k$, the reformulation is convex. Also, by defining our rate as a fraction $\frac{d_l^{t,\tau}}{d_l^{t-1,\tau-1}}$, we had availed ourselves a degree of freedom. For convenience, one may set $s_l^{t,\tau} = d_l^{t,\tau}$ for all $t$ and $\tau$ for which $t = 0$ or $\tau = 0$, in which case, the first equation becomes linear for fixed $k$:

$$\sum_l s_l^{t,0} u_l^0 + k \sum_l \sum_{1 < t' \leq t} \xi_l^{t',1} + k \sum_l \sum_{\tau \geq t} \xi_l^{1,\tau-t+1} \leq U_t.$$

REMARK 1. One may be tempted to extend Theorem 1 to the general preservation function $\pi$ and its certainty equivalence, *e.g.* to change (8) into $k\pi^{-1} \left( \mathbb{E} \left[ \pi \left( (\sum_{l,\tau} \tilde{s}_l^{t,\tau} u_l^\tau - U_t) / k \right) \right] \right) \leq 0$. Such an attempt, however, turns out to be subtly challenging. The key issue lies with the use of the multiplicative property of exp under independence: $\mathbb{E} \left[ \exp \left( \sum_{l,\tau} s_l^\tau \right) \right] = \mathbb{E} \left[ \prod_{l,\tau} \exp(s_l^\tau) \right] = \prod_{l,\tau} \mathbb{E}[\exp(s_l^\tau)]$. Replicating this for a generic preservation $\pi$ can be difficult.

The proof of Theorem 1 may seem cumbersome and technical, but mathematically, it is just a repeated application of pipeline invariance. The idea is that the preservation function $\pi(\cdot)$ preserves the functional form within the expectation, hence enabling us to evaluate $\mathbb{E}_t[\pi(\cdot)]$ repeatedly over time. In this process, it creates a nested series of relay functions, which being convex, can be represented as auxiliary variables $\xi$ in epigraph form.

This is illustrative of the concept of *pipelines*, which the stock in each grade $l$ is aligned in:

$$\mathcal{P}_l^{0,0} = \{ s_l^{0,0}, s_l^{1,1}, s_l^{2,2}, \ldots \}$$
$$\mathcal{P}_l^{0,1} = \{ s_l^{0,1}, s_l^{1,2}, s_l^{2,3}, \ldots \}$$
$$\vdots$$
$$\mathcal{P}_l^{1,0} = \{ s_l^{1,0}, s_l^{2,1}, s_l^{3,2}, \ldots \}$$
$$\mathcal{P}_l^{2,0} = \{ s_l^{2,0}, s_l^{3,1}, s_l^{4,2}, \ldots \}$$
$$\vdots$$

In the absence of any intervention, an employee belonging to a particular pipeline remains in the same pipeline across time. Attrition, here acting as the random process $\tilde{Z}$, erodes the stock in

the pipelines over time and promotion is a re-distribution mechanism across pipelines. Such an interpretation also lends some justification to the independence assumption in Theorem 1. One otherwise does not expect the pipelines to mix, save for the effect of promotions.

More specifically, in the case of manpower planning, we have the corollary:

**Corollary 1 (Binomial Pipeline)** *Under the same assumptions as Theorem 1, where $Z_l^\tau$ is binomial with success probability $q_l^\tau$ and dependence on the number of trials, the constraint*

$$k \log \left( \mathbb{E} \left[ \exp \left( (\textstyle\sum_{l,\tau} \tilde{s}_l^{t,\tau} u_l^\tau - U_t)/k \right) \right] \right) \le 0 \tag{18}$$

*may be reformulated as*

$$
\begin{aligned}
&\sum_l s_l^{t,0} u_l^0 + k \sum_{\substack{l \\ 1 < t' \le t}} \frac{s_l^{t'-1,0}}{d_l^{t'-1,0}} \xi_l^{t',1} + k \sum_{\substack{l \\ \tau \ge t}} \frac{s_l^{0,\tau-t}}{d_l^{0,\tau-t}} \xi_l^{1,\tau-t+1} \le U_t \\
&d_l^{t,\tau} \log(1 - q_l^\tau + q_l^\tau e^{u_l^\tau/k}) \le \xi_l^{t,\tau} && \forall \tau \in [M] \\
&d_l^{t',\tau} \log(1 - q_l^\tau + q_l^\tau e^{\xi_l^{t'+1,\tau+1}/d_l^{t',\tau}}) \le \xi_l^{t',\tau} && \forall t' \in [t-1], \tau \in [M-t+t']
\end{aligned}
\tag{19}
$$

*Proof.* It is easy to check that the binomial family with dependence on $x$, the number of trials, forms a pipeline invariant family with preservation function $\pi(x) = e^x$ and relay functions $\rho_l^\tau(y) = \log(1 - q_l^\tau + q_l^\tau e^y)$, where $\rho$ is convex on the whole real line for $q_l^\tau \in [0,1]$. □

The remaining challenging is to deal with the constraint associated with re-distribution across pipelines, *i.e.* the non-negative hiring constraint. Thankfully, we have:

**Proposition 1 (Re-distribution Constraint)** *Under the same assumptions as Corollary 1, for fixed $l$, the constraint*

$$k \log \left( \mathbb{E} \left[ \exp \left( \left( \textstyle\sum_\tau \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t-1,\tau-1} - d_l^{t,\tau}}{d_l^{t-1,\tau-1}} - \tilde{s}_{l+1}^{t,0} \right)/k \right) \right] \right) \le 0 \tag{20}$$

*is equivalent to the set of equations*

$$
\begin{aligned}
&-s_{l+1}^{t,0} + s_l^{t-1,0} \frac{d_l^{t-1,0} - d_l^{t,1}}{d_l^{t-1,0}} + k \sum_{1 < t' < t} \frac{s_l^{t'-1,0}}{d_l^{t'-1,0}} \xi_l^{t',1} + k \sum_{\tau \ge t-1} \frac{s_l^{0,\tau-t+1}}{d_l^{0,\tau-t+1}} \xi_l^{1,\tau-t+2} \le 0 \\
&d_l^{t-1,\tau} \log(1 - q_l^\tau + q_l^\tau e^{(1 - d_l^{t,\tau+1}/d_l^{t-1,\tau})/k}) \le \xi_l^{t-1,\tau} && \forall \tau \in [M] \\
&d_l^{t',\tau} \log(1 - q_l^\tau + q_l^\tau e^{\xi_l^{t'+1,\tau+1}/d_l^{t',\tau}}) \le \xi_l^{t',\tau} && \forall t' \in [t-2], \tau \in [M-t+t'+1]
\end{aligned}
\tag{21}
$$

The proof is technical and similar to the proof of Theorem 1; as such, it is omitted. Again, note that for fixed $k$, the problem remains convex and the simplifying assumption to set $s_l^{t,\tau} = d_l^{t,\tau}$ for all $t$ and $\tau$ where either of them is 0, keeps the first equation linear.

**Robustness and Tractability**

One may contest that the RMP model could be sensitive to the specification of the attrition estimates $q$ and that in reality, the estimates of $q$ from data over only a few years would be subject to large errors. These concerns are valid. Thankfully, the exponential disutility form of the riskiness index affords another additional benefit – its dual representation relates to the Kullback-Leibler (KL) divergence. More specifically, it is well known that:

**Proposition 2** *Let* $\mathbb{P}$ *and* $\hat{\mathbb{P}}$ *be distributions. Then*

$$k\log(\mathbb{E}_{\hat{\mathbb{P}}}e^{\tilde{Z}/k}) = \sup_{\mathbb{P}}\mathbb{E}_{\mathbb{P}}[\tilde{Z}] - k\mathcal{D}(\mathbb{P}||\hat{\mathbb{P}}) \tag{22}$$

*where* $\mathcal{D}(\mathbb{P}||\hat{\mathbb{P}})$ *is the Kullback-Leibler divergence.*

In other words, Proposition 2 indicates that each constraint, evaluated under the empirical distribution $\hat{\mathbb{P}}$, provides an upper bound for the expected violation, evaluated under the true distribution $\mathbb{Q}$, as long as the distance metric (in the KL divergence sense) between $\hat{\mathbb{P}}$ and $\mathbb{Q}$ is small. Moreover, this distance is controlled by the risk parameter $k$. We shall see how this can lead to robust outcomes later during the numerical illustration.

To solve the Risk-based Manpower Planning model, one can perform bisection search on $k$. However, it requires the proper solution of the exponential cone. In our case, the cones are of the form $f(d, \zeta; q) := d\log(1 - q + qe^{\zeta/d}) \leq \xi$, where $q \in [0, 1]$. We know that $f \sim d\log(1 - q)$ as $\zeta \to -\infty$ and $f \sim \zeta + d\log q$ as $\zeta \to \infty$. As such, the behavior of $f$ is asymptotically linear. This suggests that an approximation using a cutting plane method is likely to suffice. A sample algorithm could go along the lines of Algorithm 1.

Alternatively, one can approximate $f$ with a series of second-order cones:

**Proposition 3 (SOC Approximation)** *For* $q \in [0, 1]$*, the function*

$$f(d, \zeta; q) := d\log(1 - q + qe^{\zeta/d}) \tag{23}$$

*has a second-order cone formulation that approximates it to at least the tenth order.*

*Proof of Proposition 3.* The proof is provided in the Appendix. □

More recently, there have also been advances in the efficient computation of exponential cones, especially using interior point methods. Solvers are already available in MATLAB (CVX Research 2012) and also in Julia/JuMP (*e.g.* Miles et al. 2016), extending to MICPs, which arises when we extend the model to individuals.

---

**Algorithm 1** Cutting-Plane Algorithm for the Risk-based Manpower Planning Model

---

**Require:** Tolerances $\epsilon$, $\epsilon_{\mathcal{K}}$. Set as objective to minimize any constraint for $t = T$ (say headcount

$\sum_l s_l^{t,0} + k \sum_{l \in [L], 1 < t' \leq t} \xi_l^{t',1} + k \sum_{l \in [L], \tau \geq t} \xi_l^{1,\tau-t+1}$). Let $U_T$ be its target.

**Initialization**: $k_- \leftarrow 0$, sufficiently large $k_+$.

Let $\mathcal{R}(k)$ be the model where all exponential cone constraints $f(d, \zeta; q) \leq \xi$ in (6) are replaced

with asymptotic linear estimates, $d \log(1 - q) \leq \xi$ and $\zeta + d \log q \leq \xi$.

**while** $k_+ - k_- > \epsilon$ **do**

  $k := \bar{k} \leftarrow {(k_+ - k_-)}/2$

  Solve $\mathcal{R}(k)$. If feasible, obtain optimal value $U^*$ and optimal policy $(d^*, \zeta^*, \xi^*)$.

  **if** $\mathcal{R}(k)$ infeasible **or** $U^* > U_T$ **then**

    $k_- \leftarrow \bar{k}$

  **else**

    $\mathcal{F} \leftarrow \{f \mid f(d^*, \zeta^*; q) - \xi^* > \epsilon_{\mathcal{K}}\}$

    **if** $\mathcal{F} \neq \emptyset$ **then**

      Add hyperplane $\forall f \in \mathcal{F}$ at $(d^*, \zeta^*, \xi^*)$ into $\mathcal{R}(k)$.

    **else**

      $k_+ \leftarrow \bar{k}$

    **end if**

  **end if**

**end while**

**Output**: Optimal $k^* = k_+$ and optimal policy $(d^*, \zeta^*, \xi^*)$.

---

Our model does not compromise tractability – the number of constraints does not grow exponentially with time horizon $T$, or any of the other parameters, such as grades $L$ or maximum time-in-grade $M$. Indeed, in Theorem 1, for each $t \in [T]$, the number of exponential cone constraints required to reformulate one linear constraint is of order $O(LMT)$. Hence, in total, $O(LMT^2)$ exponential cone constraints are required. Moreover, Proposition 3 guarantees that estimating these exponential cone constraints can be done so in the same order, $O(LMT^2)$. Note that the dependence on $T$, the length of the time horizon, is only quadratic, while often behavior similar to $O(M^T)$ is observed. In practice, many solvers will exploit the redundancy in many of these constraints to vastly speed up the computation.

Lastly, we comment that one can adapt the model, by using a different $k_j$ for each constraint, indexed in a set $j \in \mathcal{J}$ and then performing a lexicographic minimization on $\boldsymbol{k} := (k_j)_{j \in \mathcal{J}}$. A general scheme of how to execute this can follow along the lines of Waltz (1967). This methodology may be employed if the decision-maker is agnostic to the relative risk aversions of each constraint and would

prefer to simply go for a 'fair' outcome in terms of the risk borne at the level of each constraint. In the interest of brevity, we do not execute the lexicographic minimization approach in this paper.

## 3. Risk-based Manpower Planning in a Firm

In this section, we illustrate the Risk-based Manpower Planning model using real data of more than $5,000$ employees in the Singapore Civil Service, who can be safely assumed to have similar job characteristics and backgrounds, tracked over 6 years. This data is collected periodically at the individualized level, and for this illustration, we are able to summarize it into the form of the inputs for our model. This includes their attrition, performance and wage patterns – personnel data that is similarly collected by most large organizations. Due to confidentiality, we are unable to reveal more about the nature of the data, though in the subsequent description, we will illustrate some features as far as we are able to share. In this illustration, we shall look at a 5-year time window, $T = 5$.

We model $L = 4$ grades in this organization, two 'individual contributor' grades labelled IC1 and IC2, which generate a large part of the productivity, and two manager grades, denoted M1 and M2. Progression occurs in this order and skipping of grades is disallowed. We truncate the maximum number of years that an employee may remain in any grade to $M = 20$, where thereafter the employee is assumed to have retired. At each grade $l$, we assume that employees are paid a base wage $\omega_l$ with an annual fixed increment $\iota_l$. Hence, $w_l^\tau = \omega_l + \tau \iota_l$. The parameters $\omega_l$ and $\iota_l$ were statistically estimated from wage data by means of a linear regression, and adjusted to make sense. Due to its sensitivity, we are unable to disclose these estimates.

We obtained performance data which varies across time-in-grade. For simplicity, we assume that the productivity of an employee can be written in the separable form, $r_l^\tau = \kappa_l \zeta^\tau$. Here, $\zeta^\tau$ is the productivity profile across time-in-grade and $\kappa_l$ is a scaling factor across different grades. The profile $\zeta^\tau$ is estimated from data and plotted in Figure 1. It rises with more years-in-grade, a reflection of the accumulation of experience, before dipping with increasing employee boredom and disengagement. Manager span of control is also assumed to follow this profile $\zeta^\tau$. Hence, $c_l^\tau = \hat{\kappa}_l \zeta^\tau$.

Retention rates $q_l^\tau$ were estimated from the data. Figure 2 illustrates the estimates. Where the data was sparse, fluctuations were severe. Nonetheless, Proposition 2 provides some loose guarantees that the estimation of constraints will remain within reasonable bounds. Later, when analyzing the robustness of the model, we shall explore this further.

Finally, we specify the constraint targets. From here on, the targets shall always be fixed as a geometric rate of growth $g$ from the initial state at time $t = 0$. We vary these rates of growth in different simulations. Table 1 summarizes this. We also require that the productivity target grows at a slightly faster rate than the headcount and budget targets.

**Figure 1**     Profile of Performance with Time-in-Grade
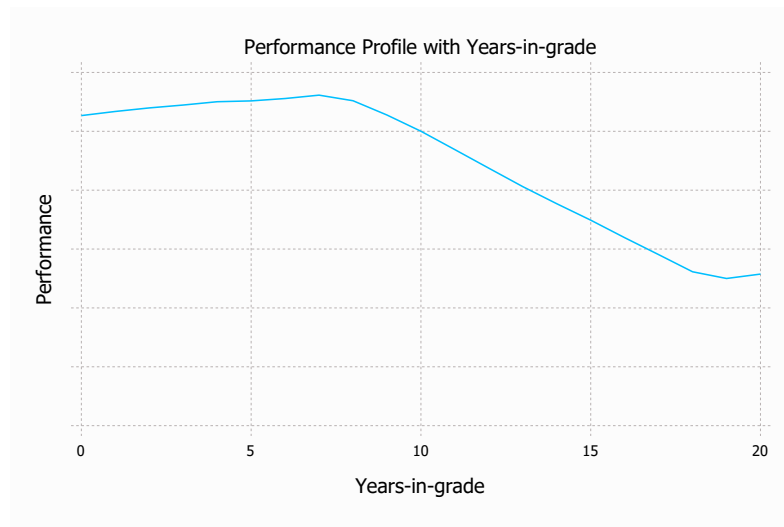


**Figure 2**     Retention Rates with Time-in-Grade
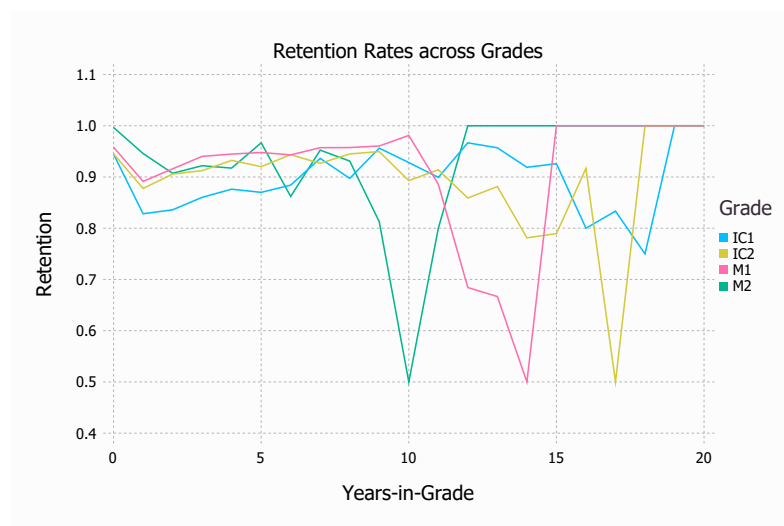


Because the certainty equivalence $C_k$ is not scale invariant, we normalized all constraints so as to ensure equitable comparisons (without having to calibrate a $\theta$ parameter for each constraint). In other words, the model penalizes the proportional violation of targets equally across constraints.

With this specification, the model seeks to minimize the risk level, $k$. It returns $k$, in addition to optimal solutions for the decision variables of hiring $h_l^t$ and promotion $d_l^{t,\tau}$. To simulate the uncertainty and test the model, for each analysis, we ran $1,000$ simulations with the random outcomes of employees' retention drawn from a binomial distribution of estimated retention rates, $q_l^\tau$, as the success probability.

**Table 1**    Specification of Constraints

| Constraint | Equation | Target | Specification |
|---|---|---|---|
| Headcount | $\sum_{l,\tau} s_l^{t,\tau} \leq H_t$ | $H_t = g_h^t H_0$ | $g_h = g$ |
| Budget | $\sum_{l,\tau} s_l^{t,\tau} w_l^{\tau} \leq B_t$ | $B_t = g_b^t B_0$ | $g_b = g$ |
| Productivity | $\sum_{l,\tau} s_l^{t,\tau} r_l^{\tau} \geq R_t$ | $R_t = g_p^t R_0$ | $g_p = 1 + 1.05(g-1)$ |

## Robustness

We first examine the robustness properties of the model. By design, the model provides guarantees against constraint violation. To illustrate this, we shall compare the risk-based model against a deterministic model. In the deterministic model, one evaluates the expected number of employees as $\mathbb{E}[s_l^{t,\tau} | \tilde{s}_l^{t-1,\tau-1}] = \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t,\tau}}{d_l^{t-1,\tau-1}} q_l^{\tau}$. Hence, in expectation,

$$\mathbb{E}[s_l^{t,\tau}] = \begin{cases} s_l^{0,\tau-t} \frac{d_l^{t,\tau}}{d_l^{0,\tau-t}} \prod_{j=\tau-t+1}^{\tau} q_l^j = d_l^{t,\tau} \prod_{j=\tau-t+1}^{\tau} q_l^j & \text{if } \tau \geq t \\ s_l^{t-\tau,0} \frac{d_l^{t,\tau}}{d_l^{t-\tau,0}} \prod_{j=1}^{\tau} q_l^j = d_l^{t,\tau} \prod_{j=1}^{\tau} q_l^j & \text{if } \tau < t \end{cases} \tag{24}$$

In the case of the highest level $l=4$, there can be no more promotions. As such, we instead obtain

$$\mathbb{E}[s_4^{t,\tau}] = \begin{cases} s_4^{0,\tau-t} \prod_{j=\tau-t+1}^{\tau} q_4^j & \text{if } \tau \geq t \\ s_4^{t-\tau,0} \prod_{j=1}^{\tau} q_4^j & \text{if } \tau < t \end{cases} \tag{25}$$

With this, we can represent $\mathbb{E}\left[\sum_{l,\tau} \tilde{s}_l^{T,\tau} u_l^{\tau}\right]$ in terms of decision variables $d_l^{t,\tau}$ and $s_l^{t,0}$:

$$\mathbb{E}\left[\sum_{l,\tau} \tilde{s}_l^{T,\tau} u_l^{\tau}\right] = \sum_{l=1}^{3} \left( s_l^{t,0} u_l^0 + \sum_{\tau=1}^{t-1} d_l^{t,\tau} u_l^{\tau} \prod_{j=1}^{\tau} q_l^j + \sum_{\tau=t}^{M} d_l^{t,\tau} u_l^{\tau} \prod_{j=\tau-t+1}^{\tau} q_l^j \right) \tag{26}$$
$$+ \left( s_4^{t,0} u_l^0 + \sum_{\tau=1}^{t-1} \tilde{s}_4^{0,\tau-t} \prod_{j=\tau-t+1}^{\tau} q_4^j + \sum_{\tau=t}^{M} \tilde{s}_4^{t-\tau,0} \prod_{j=1}^{\tau} q_4^j \right)$$

We write the deterministic model below. We shall take productivity in the last time period $(P_T)$ as the objective. Note that the deterministic model is obtained from the risk-based formulation in the limit $k \to \infty$.

$$\max \quad \mathbb{E}\left[\sum_{l,\tau} \tilde{s}_l^{T,\tau} r_l^{\tau}\right]$$

$$\text{s.t.} \quad \mathbb{E}\left[\sum_{l,\tau} \tilde{s}_l^{t,\tau}\right] \leq H_t \qquad\qquad \forall t \in [T]$$

$$\mathbb{E}\left[\sum_{l,\tau} \tilde{s}_l^{t,\tau} w_l^\tau\right] \le B_t \qquad\qquad \forall t \in [T]$$

$$\mathbb{E}\left[\sum_{l,\tau} \tilde{s}_l^{t,\tau} r_l^\tau\right] \ge P_t \qquad\qquad \forall t \in [T] \qquad\qquad (27)$$

$$\mathbb{E}\left[\sum_{\lambda,\tau} \tilde{s}_\lambda^{t,\tau} b_{l,\lambda}^\tau\right] \ge 0 \qquad\qquad \forall t \in [T], \forall l \in \mathcal{M}$$

$$\mathbb{E}\left[\sum_{\tau} \tilde{s}_l^{t-1,\tau-1} \frac{d_l^{t-1,\tau-1} - d_l^{t,\tau}}{d_l^{t-1,\tau-1}}\right] \le s_{l+1}^{t,0} \qquad\qquad \forall t \in [T], \forall l \in [L-1]$$

$$s_l^{t,0} \ge 0,\ d_l^{t,\tau} \ge 0 \qquad\qquad \forall t \in [T], \forall l \in [L], \forall \tau \in [M]$$

We first ran the model for growth rate $g = 1.02$, *i.e.* the organization is allowed to grow by 2% annually. The risk-based model seeks the minimum risk level $k^*$. In this case, $k^* \approx 35$, which yields the exponential envelope of the probability of constraint violation.

In Figure 3, we plot, for the headcount constraint, the actual materialized deviation from target $H_t - \sum_{l,\tau} s_l^{t,\tau}$ based on the uncertainty. A positive figure indicates that the headcount target was not exceeded and its magnitude gives the slack; a negative value indicates constraint violation and its magnitude, the extent. The green line represents the Markov guarantee where the probability of constraint violation should be no more than one-third. As Figure 3 illustrates, this guarantee is very loose – only 2% of the simulations exceeded this bound.

**Figure 3**     Simulated Violation of Headcount Target in Year t=1
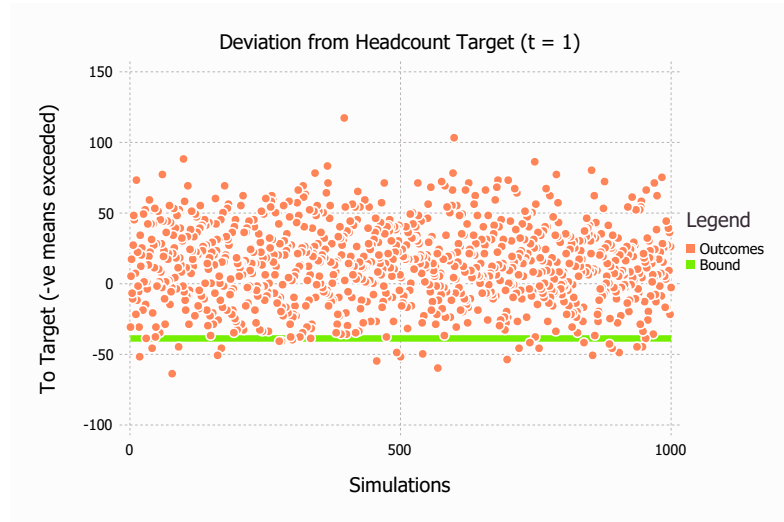


We now compare this against the deterministic model. The simulated deviations from the headcount target for each model is compared in Table 2. The risk-based model provides guarantees against constraint violation, and if violations occur, they do so with a smaller magnitude.

**Table 2**    Comparison of Constraint Violation in Risk-based and Deterministic Models

| Deviation from $H_5$ | Risk-based | Deterministic |
|---|---|---|
| Median | 8.07 | -4.40 |
| Mean | 7.18 | -5.57 |
| $1^{st}$ Quartile | -14.29 | -27.42 |

However, one can expect that the gains in the guarantees may not be universal for different specification of the targets. To illustrate this, let us vary the productivity target $P_t$ (via $g_p$), while keeping all other targets fixed. Intuitively, there should be a monotone relationship between $P_t$ and $k^*$ – the higher $P_t$, that is, the higher the productivity target that must be met, the more difficult it is to do so and hence the risk level $k^*$ of failing should be expected to rise. We try this for 3 configurations: $g_p = 1.023$ (where $k^*$ large), $g_p = 1.021$ (an intermediate region), and $g_p = 1.015$ (where $k^*$ small). Table 3 below summarizes the statistics for these 3 regimes, under a comparison between the risk-based and deterministic models.

**Table 3**    Different Regimes of Tightness of Targets

| Growth Rate (Risk level) | Tougher target $g_p = 1.023$ ($k^* \approx 232$) | | Intermediate $g_p = 1.021$ ($k^* \approx 35$) | | Easier target $g_p = 1.015$ ($k^* \approx 10$) | |
|---|---|---|---|---|---|---|
| Deviation from $P_5$ | Risk-based | Deterministic | Risk-based | Deterministic | Risk-based | Deterministic |
| Median deviation | -4.09 | 5.43 | 30.19 | 96.70 | 127.10 | 379.36 |
| Mean deviation | -5.16 | 4.14 | 29.13 | 97.63 | 123.16 | 379.07 |
| $1^{st}$ Quartile | -43.7 | -28.60 | -1.74 | 65.59 | 88.39 | 344.40 |
| Deviation from $H_5$ | Risk-based | Deterministic | Risk-based | Deterministic | Risk-based | Deterministic |
| Median deviation | 17.95 | 15.81 | 41.64 | 18.39 | 111.18 | 14.51 |
| Mean deviation | 18.91 | 16.59 | 42.02 | 18.45 | 113.48 | 13.92 |
| $1^{st}$ Quartile | -14.30 | -14.19 | 12.64 | -12.61 | 82.18 | -16.49 |

In Table 3, we compare two constraints. The first is the productivity constraint at time $T$. This was the objective in the deterministic model and hence we should reasonably expect the deterministic model to out-perform the risk-based model in all instances. Their difference can be understood as the cost of having guarantees against constraint violations. In the second, we compare the headcount constraint. Across the different regimes, since the deterministic model is agnostic to the risk of constraint violation, the distribution in the deviation from the headcount target is approximately the same. Here, however, we can see the action of the risk-based model.

We observe three kinds of scenarios:

1. When $k^*$ is very large (first regime), the problem is near infeasible. In this case, the guarantees on constraint violation erode away and the risk-based model approximates the deterministic model. The guarantees are so minimal, it is effectively sub-optimal. In other words, when the system is near its limits of operability, robustness is a luxury that cannot be afforded.

2. When $k^*$ is very small, we are in the third regime. Here, the guarantees are very sharp – so sharp it is over-conservative. As seen in Table 3, the loss in productivity is sizeable. On the other hand, the deterministic model is not without reproach – a huge trade-off between headcount and productivity was made, just by virtue of the fact that productivity was the objective. A reasonable course of action at this point is to tighten the target.

3. There is an intermediate region where the trade-off is balanced to some extent. In the second regime, the risk-based model does not incur a large cost to productivity, yet provides reasonable guarantees against constraint violation.

In the last segment of this analysis on robustness, we examine if the optimal solution is robust to the input parameters of attrition rates. To do so, we smooth the attrition rates (one minus the retention rates in Figure 2) using a Loess regression and prune negative values. The smoothed attrition rates are shown in Figure 4. Here, points represent the raw estimates and lines the smoothed outcome. We then perform the same analysis we have done before.

**Figure 4**    Loess Smoothed Attrition Rates for each Grade
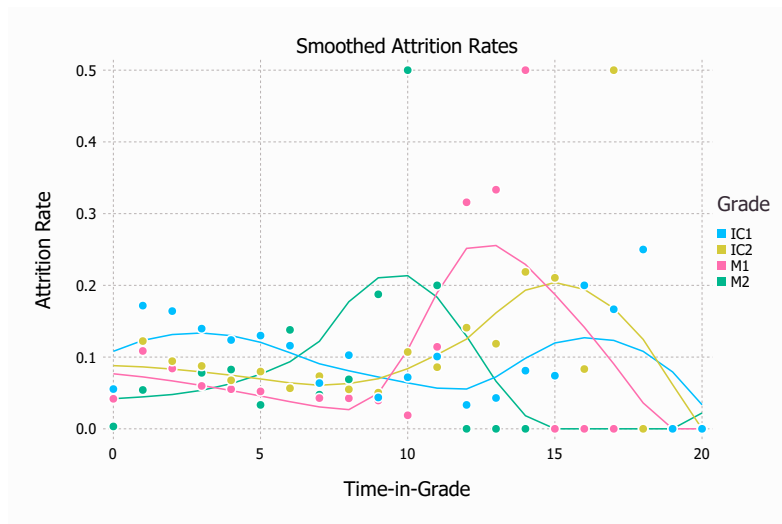


With smoothing, the risk level rises to $k^* \approx 44$ from the previous $k^* \approx 35$. Additionally, we also examine the optimal policies for promotion (in Figure 5 which can be compared against the optimal policy without smoothing in Figure 7) and hiring (in Figure 6 where the original policy is in points and the smoothed version is lined). We can see that there are only slight differences between the optimal policies suggested by the two models.

**Figure 5** Policy for Progressing from IC1 to IC2 under Smoothing
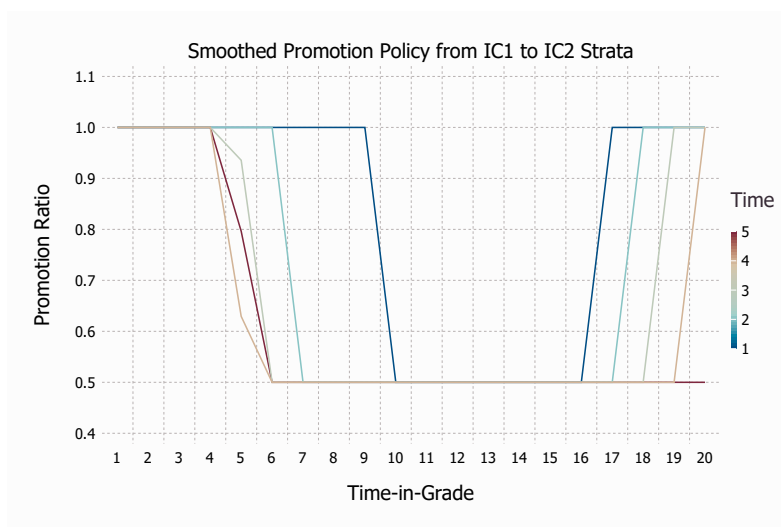


**Figure 6** Policy for Hiring across Grade and Time under Smoothing
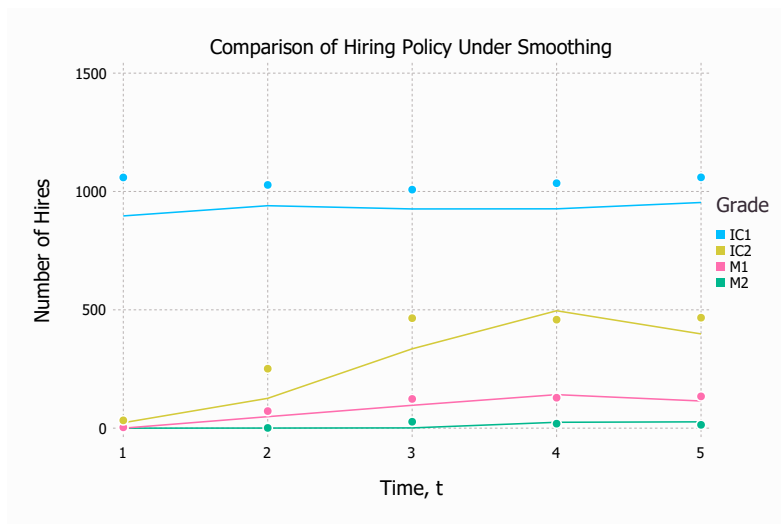


### Insights into HR

In this section, we examine insights that can be gleaned for HR. In the first instance, we are interested in the question: When is it optimal to promote employees? In other words, how long should I keep an employee at a particular grade before promoting him/her?

We study the *promotion ratio* $d_l^{t,\tau}/d_l^{t-1,\tau-1}$, the proportion of employees at time $t-1$ whom we are choosing to retain at grade $l$ for an additional year, having already spent $\tau-1$ years at this grade $l$. The remainder are promoted (or released if $h_{l+1}^t < 0$). The closer this ratio is to 1, the fewer employees we are promoting. To keep the solutions reasonable, we have constrained in the model that $d_l^{t,\tau}/d_l^{t-1,\tau-1} \geq 0.5$, that is, that no more than half of existing employees can be promoted in

any particular year. Figure 7 shows the policy for progressing employees at grade IC1 to grade IC2 as prescribed by the Risk-based Manpower Planning model.

**Figure 7**     Policy for Progressing from IC1 to IC2 at Optimal Solution
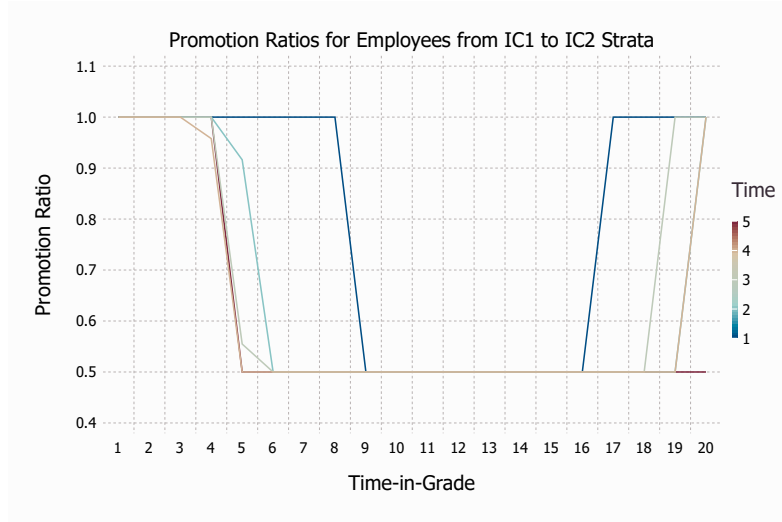


The prescribed policy is a threshold – the model believes that employees should not progress to the next grade until they have accumulated a minimum number of years, after which, they should be promoted with haste. There is a certain logic in this. In the early years, the productivity of employees rises with time spent in that grade due to the learning curve (Figure 1). As such, promoting employees too early incurs an opportunity cost of potential productivity. The model avoids this. After some point, remaining for too long at the same grade can have a disengaging effect on employees and they may leave the organization (Figure 2). The model also avoids this, by expediting their promotion after some time. In other words, the model seeks a balance between the productivity an employee brings, and the risk of losing the employee. This finding lends numerical support not just to the choice of 'time-based progression', but also its rationale.

In this second piece of analysis, we shall examine the impact that the growth rate $g$ has on the optimal risk level $k^*$. As before, we fixed the allowed growth rates of headcount, budget and productivity to be a function of $g$. Now we vary $g$. Figure 8 plots the relationship.

From Figure 8, we infer that there is a higher risk level when the growth rate is smaller. This mirrors common wisdom that it is easier to grow firms than to downsize. The explanation the model gives is this: Risk originates from the uncertainty – resignations. The higher the growth rate $g$, the greater the number of new recruits. Recruitment of employees fills the vacancies created by those who left and thus mitigates the uncertainty. As such, the more employees that can be recruited, the larger the capacity of HR to manage the risks arising from resignation, and thus the lower risk on the overall.

**Figure 8**    Risk Level $k^*$ at Different Growth Rates $g$
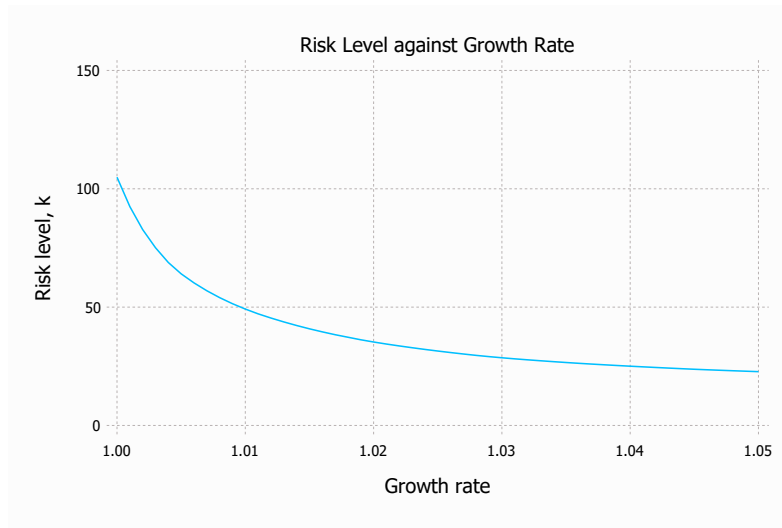


The simple consequence of this is that there are inherent operational risks to a lack of organizational renewal! Yet this is not necessarily a straightforward question to address. For example, in an organization with a higher attrition level, we can expect two competing forces at work. One, the higher the attrition, the greater the uncertainty and hence the higher the risk. Two, the higher the attrition, the greater the capacity to hire since there are more vacancies to replace, hence the lower the risk. We study which effect really plays out in our dataset.

In our model, $q_l^\tau$ represents the retention rate of officers having spent time $\tau$ at grade $l$. Hence, the attrition rate is $\alpha_l^\tau = 1 - q_l^\tau$. We now artificially suppress or boost the attrition rate by a factor of $a$, via $\bar{\alpha}_l^\tau = a \cdot \alpha_l^\tau$. If $a < 1$, the attrition rate is suppressed, and vice versa. As such, we have a new $\bar{q}_l^\tau = 1 - \bar{\alpha}_l^\tau$.

Figure 9 plots what happens to the risk level $k^*$ as we vary $a$. With lower attrition, the risk level $k^*$ rises. This is a grim consequence for advanced economies where an ageing population is beginning to take hold. As older employees are often less employable in the workforce, they tend to move between organizations less frequently than younger employees. As such, with ageing population, firms can expect to see attrition rates fall across the board. Instead, they will be faced with ever rising challenges in managing their workforce. This is not to mention that the shrinking workforce would force many firms to reduce their growth rates, which further heightens the risk.

For the final insight, we look at varying the tightness of a constraint. The inherent difficulty of public sector organizations to lay off their employees has often been quoted as a challenge to managing their workforce. To elucidate this, we perform the following analysis. Recall that we could calibrate $\theta$ in $C_{k,\theta}$ to dictate the tightness the bounds required for the corresponding constraint. Now, we do so for the hiring constraint. The lower the value of $\theta$, the more averse the model is to releasing employees. Figures 10 and 11 tell us the consequences of this.

**Figure 9** Risk Level $k^*$ with Different Scaling of Attrition $a$
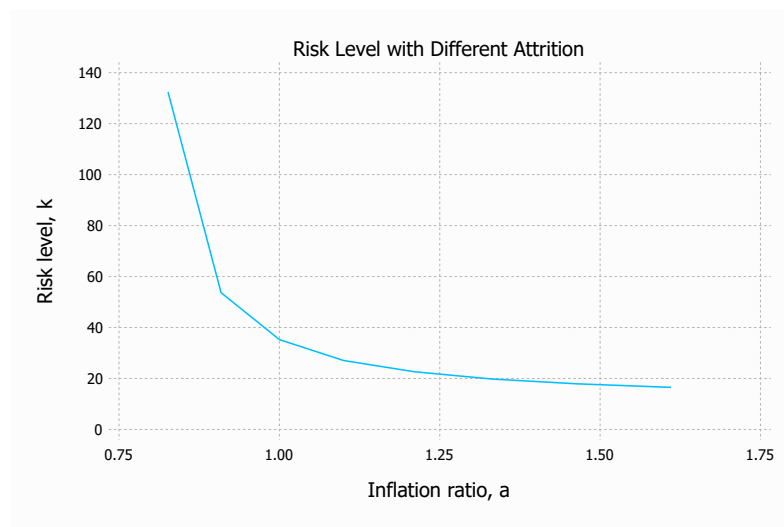


**Figure 10** Risk Level $k^*$ with Tightening of Hiring Constraint
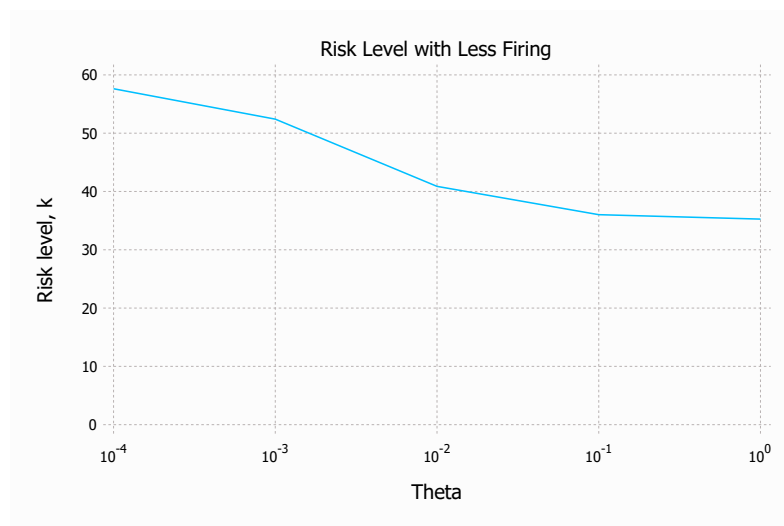


In Figure 10, we can see that the difference between not allowing any firing and allowing some firing is an almost doubling of the risk level. Figure 11 illustrates the trade-off. We plot here the largest number of employees released amongst the $1,000$ simulations. If this number is negative, it means that in all the simulations, there wasn't a single case where an employee was released. At risk level $k^* \approx 35$ and $\theta = 1$, about 40 employees were released in total across the grades. If the decision-maker is to refrain from releasing any of these employees, then $\theta$ must be decreased to $10^{-3}$. This would incur an almost 50% increase in the risk level.

This quantifies the natural challenges faced by PSOs compared to their private sector counterpartds. In this regard, it is therefore paramount that PSOs find new and innovative ways to rejuvenate and renew their workforce.

**Figure 11** Largest Number of Employees Released in any of the $1,000$ Simulations
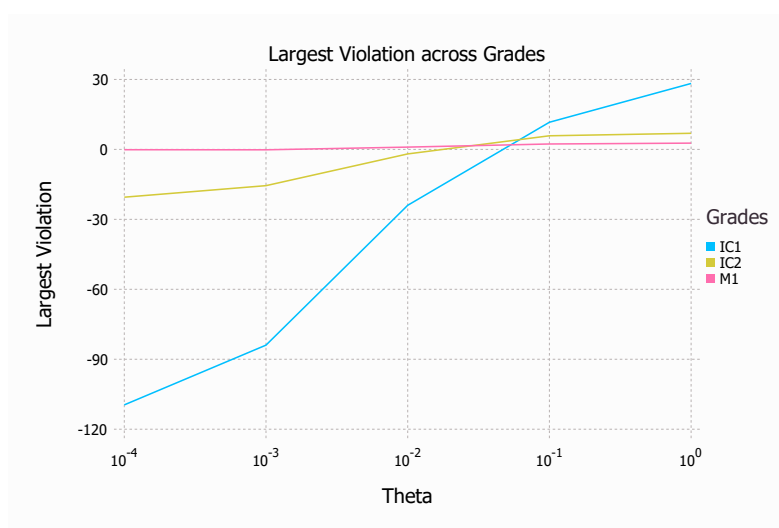


## 4. Conclusions

We have presented a tractable model for manpower planning. While the context is described within a PSO, the model can be utilized in certain job functions in profit-seeking firms. We have also illustrated HR insights that can be applicable to firms, such as the need for organizational renewal.

At its root, the Risk-based Manpower Planning model is an application of the concept of pipeline invariance under the context of multi-period optimization. The general intuition is that while it is difficult to perform multi-period optimization, we may alleviate these difficulties if we declare a formal structure (here, pipeline invariance) on how the decisions and the uncertainty are related, and hence exploit this structure to gain tractable formulations. On this note, we hope, in the future, to construct a formal framework for using pipeline invariance in multi-period optimization problems.

## Acknowledgements

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## A. Proof of Proposition 3

*Proof.* First, we show that our constraints of the form $f(d, \zeta; q) = d \log(1 - q + q e^{\zeta/d}) \leq \xi$ can be written in terms of exponential cones $\exp(\frac{x}{z}) \leq \frac{y}{z}$, where $y \geq 0$ and $z > 0$. There is a subtlety about $z > 0$ in the sense that we will take the closure of this set later, so we can allow $z \geq 0$ eventually.

Now, with some manipulation, our constraint is equivalent to

$$(1-q)e^{-\xi/d} + q e^{(\zeta-\xi)/d} \leq 1 \tag{28}$$

Notice that $de^{-\xi/d}$ is jointly convex in both $d$ and $\xi$ as it is a perspective map. As such, we could have replaced it with its epigraph formulation by introducing an auxiliary variable. The same could be done for the $de^{(\zeta-\xi)/d}$. As such, (28) has the re-formulation:

$$(1-q)y_1 + q y_2 \leq d$$
$$e^{-\xi/d} \leq y_1/d \qquad\qquad\qquad y_1 \geq 0 \tag{29}$$
$$e^{(\zeta-\xi)/d} \leq y_2/d \qquad\qquad\qquad y_2 \geq 0$$

In the second part of the proof, we show that any exponential cone of the form $\exp(\frac{x}{z}) \leq \frac{y}{z}$, where $y \geq 0$ and $z > 0$ has a second-order cone approximation at least to the tenth order. It suffices to show that $e^x$ can be expressed as a sum of squares of polynomials up to the tenth order. Indeed, one can check that the following expansion holds:

$$e^x = \frac{1}{10!}(x+1)^{10} + \frac{1}{2 \cdot 8!}\left(x + \frac{5}{3}\right)^8 + \frac{7 \cdot 19}{9!}\left(x + \frac{1963}{855}\right)^6 + \frac{3 \cdot 179369}{5^2 \cdot 10!}\left(x + \frac{14417}{4958}\right)^4 \tag{30}$$
$$+ \frac{37 \cdot 130853761}{5 \cdot 13 \cdot 5281 \cdot 9!}\left(x + \frac{12929}{3684}\right)^2 + \frac{2 \cdot 27697}{13^2 \cdot 47 \cdot 163} + O(x^{11})$$

$\square$

REMARK 2. Curiously, $e^x$ cannot be expanded in this manner to arbitrarily high order. Indeed, this expansion fails to produce positive coefficients starting from the 72nd order.