Analysis of Probabilistic Combinatorial Optimization Problems in Euclidean Spaces*

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Abstract

Probabilistic combinatorial optimization problems are generalized versions of deterministic combinatorial optimization problems with explicit inclusion of probabilistic elements in the problem definitions. Based on the probabilistic traveling salesman problem (PTSP) and on the probabilistic minimum spanning tree problem (PMSTP), the objective of this paper is to give a *rigorous* treatment of the probabilistic analysis of these problems in the plane. More specifically we present general finite-size bounds and limit theorems for the objective functions of the PTSP and PMSTP. We also discuss the practical implications of these results and indicate some open problems.

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1 Introduction

During the last decade combinatorial optimization has undoubtedly been one of the fastest growing and most exciting areas in the field of discrete mathematics. Needless to say, the related scientific literature has been expanding at a very rapid pace. An example of particular relevance to this paper is the excellent review volume on the traveling salesman problem (TSP) [10].

This paper is concerned with a specific family of combinatorial optimization problems whose common characteristic is the explicit inclusion of probabilistic elements in the problem definitions, as will be explained in Section 2. For this reason we shall refer to them as probabilistic combinatorial optimization problems (PCOPs). The analysis of these problems was initiated in [5] with the traveling salesman problem (see also [7]) and since then has been extended to the vehicle routing problem in [8], the shortest path problem in [6], the spanning tree problem and the traveling salesman facility location problem in [2]. There are several motivations for investigating the effect of including probabilistic elements in combinatorial optimization problems: among them two are particularly important. The first one is the desire to formulate and analyze models which are more appropriate for real-world problems where randomness is present. There are many important and interesting applications of PCOPs, especially in the context of strategic planning, communication systems, job scheduling, etc. For a detailed description of such problems the reader is refered to [2, 5, 7, 8]. The second motivation is an attempt to analyze the robustness (with respect to optimality) of optimal solutions for deterministic problems, when the instances for which these problems have been solved, are modified.

One can also introduce the study of PCOPs in the general framework of a priori optimization versus re-optimization strategy (see [3]). In many applications, one finds that, after solving a given instance of a combinatorial optimization problem, it becomes necessary to solve repeatedly many other instances of the same problem. These other instances are usually just variations of the instance solved originally. The most obvious approach in dealing with such cases is to attempt to solve optimally every potential instance of the original problem. Throughout the paper, we call this approach the re-optimization strategy. Rather than re-optimizing every potential instance, a different strategy would be to find an a priori solution to the original problem and then update this a priori solution to answer each particular instance/variation. The PCOPs correspond to such an alternative strategy.

Based on the probabilistic traveling salesman problem (PTSP) and on the probabilistic minimum spanning tree problem (PMSTP), the objective of this paper is to give a *rigorous* treatment of the probabilistic analysis of these problems in the plane. More specifically we present general finite-size bounds and limit theorems for the objective functions of the PTSP and PMSTP. In addition to their own theoretical interests, the importance of these results comes from their algorithmic applications. In order to justify this affirmation, let us review the case of the traveling salesman problem. In [1] it has been shown that, for any infinite sequence of bounded independent and identically distributed random variables $(X_i)_i$ with values in $\mathbf{R}^2$, the length of the shortest tour through $(X_1, \ldots, X_n)$ is asymptotic to $\beta_{tsp}\sqrt{n}$ with probability one. This theoretical result has become widely recognized to be at the heart of the probabilistic evaluation of the performance of heuristic algorithms for vehicle routing problems. In fact it was used as the main argument in the probabilistic analysis of partitionning algorithms for the TSP in [9]. In [16] it was mentioned that, in order to rigorize the result contained in [9], complete convergence was necessary instead of the almost sure convergence of [1]; the complete convergence for the TSP functional was then proved in [14].

After giving the necessary definitions and notations in Section 2, we briefly summarize the PTSP results of [5] in Section 3. The main interest of the section is to give in details the (unpublished) proof of the asymptotic convergence for the PTSP, using a general result of [13] about subadditive functionals. Section 4 is the principal section of the paper and contains a full discussion of the PMSTP results. In Subsection 4.1, we first compare the PMSTP with its deterministic special case, the MSTP. We then evaluate, in Subsection 4.2, the variations of the PMSTP functional due to two perturbations: first, a change in the probability of presence of a point, and second, a deletion of a point. Based on these two preliminary subsections, we derive, in Subsection 4.3, upper and lower bounds for finite size PMSTP in the square $[0,1]^2$. We then present, in Subsection 4.4 the analysis of the asymptotic behavior of the PMSTP under the assumption of independent and uniformly distributed points in the square $[0,1]^2$. The fact that the PMSTP functional is not monotone makes it necessary to develop specific techniques in order to obtain such a result. Finally, in Section 5, we derive, for comparison with the PTSP and the PMSTP, the asymptotic behavior of the alternative strategy of re-optimizing the problems. This section rigorizes a result about the PTSP contained in [5] and shows that complete convergence of the TSP and MSTP functionals are necessary in order to derive the asymptotic analysis of the re-optimization strategy. In the last section, we study generalizations and we list some important open problems.

2 Definitions and Notations

We consider sets of points in Euclidean space $\mathbf{R}^2$, assuming distances between points to be the ordinary Euclidean distance, hereafter written d. For a given finite set of points, the traveling salesman problem (TSP) consists of finding a tour through the points of minimum total length and the minimum spanning tree problem (MSTP) consists of finding a spanning tree of minimum total length. We define the following general probabilistic version of these two problems:

The Probabilistic Traveling Salesman Problem (PTSP):

Consider a problem of routing through a set V of n points. On any given instance of the problem, only a random subset of points (chosen according to a probability law defined on the power set 2^V of V) has to be visited. We wish to find a priori a tour through all n points. On any given instance, the subset of points present will then be visited in the *same order* as they appear in the a priori tour. The problem of finding such a tour of minimum expected length under this skipping strategy is defined as a Probabilistic Traveling Salesman Problem.

The Probabilistic Minimum Spanning Tree Problem (PMSTP):
Given a set V of n points, only a random subset of points (chosen according to a probability law defined on the power set 2^V of V) is present on any particular instance of the problem. We wish to find a priori a spanning tree through all the points so that, for any subsequent random subset of points, the tree is retraced deleting the points that are not present (with their adjacent edges), provided the deletion does not disconnect the tree (note that, with this strategy, there can be points which will not be present but still kept in the tree; the "disconnecting" quality of a point is a global property and depends upon the presence or non-presence of other points. This is in contrast with the PTSP in which the "disconnecting" quality of a point is a local property). The problem of finding an a priori spanning tree of minimum expected length is the Probabilistic Minimum Spanning Tree Problem.

In this paper, we will concentrate on the special case for which each point has a probability p of being present, independently of the others. The detailed notations and assumptions are the following: $(x_i)_i = (x_1, x_2, \ldots)$ represents an arbitrary infinite sequence of points in $\mathbf{R}^2$; $x^{(n)} = (x_1, x_2, \ldots, x_n)$ are the first n points of $(x_i)_i$. If the positions of the points are random, the sequence will be denoted by upper-case letters, i.e., $(X_i)_i = (X_1, X_2, \ldots)$. Associated with $(x_i)_i$ is an infinite sequence $(Y_i)_i$ of i.i.d. Bernouilli random variables with parameter p such that point x_j is present if and only if $Y_j = 1$. The different functionals of interest are:

$L_{tsp}(x^{(n)})$: the length of an optimal TSP tour through $x^{(n)}$,
$L_{mstp}(x^{(n)})$: the length of an optimal MSTP tree through $x^{(n)}$,
$El_p^{ptsp}(x^{(n)})$: the expected length, in the PTSP sense, of an optimal PTSP tour through $x^{(n)}$,
$El_p^{pmstp}(x^{(n)})$: the expected length, in the PMSTP sense, of an optimal PMSTP tree through $x^{(n)}$,
$O_p^{ptsp}(x^{(n)})$: the expected length computed under the strategy of re-optimizing the tour for each instance of the problem through $x^{(n)}$,
$O_p^{pmstp}(x^{(n)})$: the expected length computed under the strategy of re-optimizing the tree for each instance of the problem through $x^{(n)}$.

When the positions of the points are random the previous quantities are all random variables and their expectations (with respect to the position of the points) are noted $\mathbf{E}[.]$.

3 Probabilistic Traveling Salesman Problems

3.1 Background

The following result (given here for comparison with the TSP) gives the objective function of the PTSP and can be found in [5, 7] :

Fact 3.1 *The expected length of any tour $t = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, x_{\sigma(1)})$ through $x^{(n)}$ is given by $p^2 \sum_{r=0}^{n-2} \sum_{i=1}^{n} (1-p)^r d(x_{\sigma(i)}, x_{\sigma((i+r \bmod n)+1)})$.*

In the next result, we consider a sequence of points in $[0,t]^2$ and give upper and lower bounds on the expected length of an optimal PTSP tour through the first n points of the sequence. The proofs of these bounds can be found in [5] and are not reproduced here (the upper bounds are obtained via special tour-constructions, the arguments being very much similar to the ones developed in details for the PMSTP in Section 4.3). The bounds will be useful in the asymptotic analysis of Section 3.2.

Lemma 3.1 *(i) Let $(x_i)_i$ be an arbitrary sequence of points in $[0,t]^2$ and p be the probability of existence of each point, then:*

$$El_p^{ptsp}(x^{(n)}) \leq \begin{cases} (\sqrt{2(np-2)} + 13/4)t & \text{if } np \geq 2.5, \\ (np/2+3)t & \text{otherwise.} \end{cases} \tag{3.1}$$

(ii) Let $(X_i)_i$ be a sequence of points independently and uniformly distributed over $[0,t]^2$ and p be the probability of existence of each point, then:

$$\mathbf{E}[El_p^{ptsp}(X^{(n)})] \leq \begin{cases} (\sqrt{(4/3)(np-3)} + 11/12 + \sqrt{2})t & \text{if } np \geq 3.75, \\ (np/3 + 2/3 + \sqrt{2})t & \text{otherwise.} \end{cases} \tag{3.2}$$

Under the same conditions we have:

$$\mathbf{E}[El_p^{ptsp}(X^{(n)})] \geq ((5/8)p\sqrt{n}(1 - \sqrt{n}(1-p)^{n-1}))t. \tag{3.3}$$

3.2 Asymptotic Analysis

The objective of this asymptotic analysis is to obtain a *strong* limit law for the PTSP.

Theorem 3.1 *Let $(X_i)_i$ be an infinite sequence of points independently and uniformly distributed over $[0,1]^2$ and p be the probability of existence of each point. Then there exists a positive constant $c(p)$ such that:*

$$\lim_{n\to\infty} \frac{El_p^{ptsp}(X^{(n)})}{\sqrt{n}} = c(p) \text{ (a.s.)}. \tag{3.4}$$

Proof:
We will use a general result about subadditive functionals obtained in [13]. Before stating this result, let us give some definitions. By a functional Φ we mean a real-valued function of the finite subsets of $\mathbf{R}^2$. We say that (a) Φ is *euclidean* if it is linear and invariant under translation; (b) Φ is *monotone* if $\Phi(y \cup A) \geq \Phi(A)$ for any y in $\mathbf{R}^2$ and finite subsets A of $\mathbf{R}^2$; (c) Φ is *bounded* if $var[\Phi(X^{(n)})] < \infty$ whenever the points of $X^{(n)}$ are independent and uniformly distributed in $[0,1]^2$; (d) Φ is *subadditive* if whenever $(Q_i)_{1 \leq i \leq m^2}$ is a partition of the square $[0,t]^2$ into squares with edges parallel to the axle and of length t/m, and whenever $(x_i)_i$ is an arbitrary sequence of points in $\mathbf{R}^2$, then there exists a positive constant B such that $\Phi(x^{(n)} \cap [0,t]^2) \leq \sum_{i=1}^{m^2} \Phi(x^{(n)} \cap Q_i) + Btm$. In [13], the author proves the following result:

Theorem (Steele): *Let Φ be a subadditive, euclidean, monotone and bounded functional. If $(X_i)_i$ is a sequence of points independently and uniformly distributed over $[0,1]^2$, then there exists a constant φ such that $\Phi(X^{(n)})/\sqrt{n}$ goes to φ almost surely when n goes to infinity.*

It is not difficult to verify that the functional $El_p^{ptsp}(X^{(n)})$ is euclidean, monotone and bounded. The most demanding is to show that it is subadditive and this will be the purpose of the remaining part of this proof.

Suppose that the infinite sequence $(x_i)_i$ is contained in $[0,t]^2$ and consider the following tour through the sequence $x^{(n)}$ in $[0,t]^2$: first construct the optimal PTSP tours through $x^{(n)} \cap Q_i$ for $1 \leq i \leq m^2$. Then , in each square Q_i where $x^{(n)} \cap Q_i$ is not empty, choose one point as a representative and consider it as *always* present; finally construct a TSP tour through the set S of all representatives (at most m^2 points). The combination of the small PTSP subtours together with this TSP tour gives a spanning walk through $x^{(n)}$.

By the fact that the representatives are always present, this spanning walk has an expected length (in the PTSP sense) given by:

$$\sum_{i=1}^{m^2} \psi_p^{ptsp}(x^{(n)} \cap Q_i) + L_{tsp}(S), \tag{3.5}$$

where $\psi_p^{ptsp}(x^{(n)} \cap Q_i)$ denotes the new expected length (computed under the assumption that the representative is always present) of the PTSP tour initially constructed in Q_i, and where $L_{tsp}(S)$ is the length of the TSP tour through the set of representatives S.

One can then delete some arcs and transform this spanning walk into a tour of smaller expected length. The expected length of this tour decreases if one turns back each representative into a normal point (i.e., a point that is present with a probability p); thus we obtain a tour through $x^{(n)}$ of expected length bounded from above by (3.5) and from below by $El_p^{ptsp}(x^{(n)})$. Finally from (3.1) applied with $p = 1$, we have $L_{tsp}(S) \leq btm$ for an appropriate constant b. All this together implies that:

$$El_p^{ptsp}(x^{(n)}) \leq \sum_{i=1}^{m^2} \psi_p^{ptsp}(x^{(n)} \cap Q_i) + btm. \tag{3.6}$$

Now, for all $1 \leq i \leq m^2$, we have:

$$\psi_p^{ptsp}(x^{(n)} \cap Q_i) \leq El_p^{ptsp}(x^{(n)} \cap Q_i) + 2(\sqrt{2}t/m)(1-p), \tag{3.7}$$

since the difference between the two types of expected length arises only when the point playing the representative is not present (with probability $1-p$), this difference being then no more than twice the diagonal of the small square. Finally from (3.6) and (3.7) we get:

$$El_p^{ptsp}(x^{(n)}) \leq \sum_{i=1}^{m^2} El_p^{ptsp}(x^{(n)} \cap Q_i) + (2\sqrt{2}(1-p) + b)tm, \tag{3.8}$$

which shows that the PTSP functional is subadditive.

∎

From (3.1), we know that $El_p^{ptsp}(X^{(n)})/\sqrt{n}$ is uniformly bounded by a constant (depending only on p), so Theorem 3.1 and Lebesgue's dominated convergence theorem imply the following result.

Corollary 3.1 *Under the conditions of Theorem 3.1 we have:*

$$\lim_{n\to\infty} \frac{\mathbf{E}[El_p^{ptsp}(X^{(n)})]}{\sqrt{n}} = c(p). \tag{3.9}$$

Remark:
Theorem 3.1 proves the existence of a constant without giving details on its value; in fact, for all similar asymptotic results, the respective limiting constants are unknown and only bounds have been established. Our problem is no exception and one can use (3.2) and (3.3) to get:

$$5p/8 \leq c(p) \leq \sqrt{4p/3}. \tag{3.10}$$

4 Probabilistic Minimum Spanning Tree Problems

4.1 Relationships between the PMSTP and the MSTP

The objective function of a feasible solution to the PMSTP is given by the following result.

Property 4.1 *Let $(x_i)_i$ be an arbitrary sequence of points and p be the probability of existence of each point. The expected length (in the PMSTP sense) of any tree T through $x^{(n)}$, defined by the set of its edges A_T, is given by*

$$El_T = \sum_{e\in A_T} \beta_T(e)d(e), \tag{4.1}$$

where

$$\beta_T(e) = \sum_{e \in A_T} (1-(1-p)^{|V_T(e)|})(1-(1-p)^{n-|V_T(e)|})d(e),$$

with $d(e)$ the length of edge e, and $V_T(e)$ the subset of points contained in one of the two subtrees obtained from T by removing the edge e.

Proof:
Let e be an edge of T. For any instance of the problem, e will be present if and only if there is at least one point present in $V_T(e)$ *and* if there is at least one point present in $x^{(n)} \setminus V_T(e)$.

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The next result gives bounds on the weight $\beta_T(e)$ of any edge e of a tree T.

Property 4.2 *Let $(x_i)_i$ be an arbitrary sequence of points and p be the probability of existence of each point. For any edge e of a tree T through $x^{(n)}$ we have*

$$p(1-(1-p)^{n-1}) \leq \beta_T(e) \leq (1-(1-p)^{\lceil n/2 \rceil})^2. \tag{4.2}$$

Proof:
This follows from the fact that for $p \in]0,1[$ the function $f(x) = (1-(1-p)^x)(1-(1-p)^{n-x})$ is monotone increasing on $[0, n/2]$ (note also that $\beta_T(e) = 0$ if $p = 0$, and $\beta_T(e) = 1$ if $p = 1$).

∎

The principal result of this section gives the following relationships between the objective functions of the PMSTP and MSTP.

Lemma 4.1 *Let $(x_i)_i$ be an arbitrary sequence of points and p be the probability of existence of each point, then*

$$p(1-(1-p)^{n-1})L_{mstp}(x^{(n)}) \leq El_p^{pmstp}(x^{(n)}) \leq (1-(1-p)^{\lceil n/2 \rceil})^2 L_{mstp}(x^{(n)}). \tag{4.3}$$

Proof:
Let L_T be the length of any tree through $x^{(n)}$. From Property 4.1 and Property 4.2 we have

$$p(1-(1-p)^{n-1})L_T \leq El_T \leq (1-(1-p)^{\lceil n/2 \rceil})^2 L_T. \tag{4.4}$$

Let T^* be an optimal PMSTP tree. From (4.4) we have

$$El_p^{pmstp}(x^{(n)}) = El_{T^*} \geq p(1-(1-p)^{n-1})L_{T^*},$$

and since, by definition, $L_{mstp}(x^{(n)}) \leq L_{T^*}$, we obtain the lower bound of (4.3). Let T^{**} be an optimal MSTP tree. From (4.4) we have

$$El_{T^{**}} \leq (1-(1-p)^{\lceil n/2 \rceil})^2 L_{T^{**}} = (1-(1-p)^{\lceil n/2 \rceil})^2 L_{mstp}(x^{(n)}),$$

and since, by definition, $El_p^{pmstp}(x^{(n)}) \leq El_{T^{**}}$, we obtain the upper bound (4.3).

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4.2 Analysis of Two Perturbations on Trees

In the following result, we bound the variation occuring in the objective function of a tree when one of the leaves is chosen to be always present.

Lemma 4.2 *Let $(x_i)_i$ be an arbitrary sequence of points in a bounded set A, and p be the probability of existence of each point. For any tree T through $x^{(n)}$, choose from $x^{(n)}$ a leaf, say x_i, and consider it as always present, and let $E_i l_T$ be the new expected length. Then we have*

$$El_T \leq E_i l_T \leq El_T + \delta(A)(1-(1-p)^{n-1})^2(1-p)/p, \tag{4.5}$$

where $\delta(A)$ denotes the diameter of the set A.

Proof:
Without lost of generality, suppose that for any edge e of the tree T, $V_T(e)$ (see Property 4.1) is chosen to be the subset of points that does not contain x_i. We then have

$$E_i l_T = \sum_{e \in A_T} (1-(1-p)^{|V_T(e)|}) d(e). \tag{4.6}$$

From (4.1) and (4.6) we then have

$$\begin{aligned} E_i l_T - El_T &= \sum_{e \in A_T} (1-(1-p)^{|V_T(e)|})(1-p)^{n-|V_T(e)|} d(e) \\ &\leq (1-(1-p)^{n-1}) \sum_{e \in A_T} (1-p)^{n-|V_T(e)|} d(e) \\ &\leq \delta(A)(1-(1-p)^{n-1}) \sum_{e \in A_T} (1-p)^{n-|V_T(e)|}. \end{aligned} \tag{4.7}$$

Let us now prove that, for any tree T with a leaf that is always present, we have

$$\sum_{e \in A_T} (1-p)^{n-|V_T(e)|} \leq (1-(1-p)^{n-1})(1-p)/p, \tag{4.8}$$

where, as before, $V_T(e)$ is chosen to be the subset of points that does not contain the leaf. Let us prove it by induction. For $n = 2$, (4.8) is true. Let suppose it to be true up to $n = k-1$, and consider a tree T through $x^{(k)}$, and suppose x_i ($1 \leq i \leq k$) is a leaf that is always present. Suppose x_j is another leaf of T and let T' be the tree obtained from T by removing x_j and its adjacent edge, say e_j. Now, for any edge e of T', if $x_j \in V_T(e)$ then $|V_{T'}(e)| = |V_T(e)| - 1$, else $|V_{T'}(e)| = |V_T(e)|$. This implies that

$$\begin{aligned} \sum_{e \in A_T} (1-p)^{k-|V_T(e)|} &= (1-p)^{k-|V_T(e_j)|} + \sum_{e \in A_{T'}} (1-p)^{k-|V_T(e)|} \\ &= (1-p)^{k-1} + \sum_{e \in A_{T'}} (1-p)^{k-|V_T(e)|} \end{aligned}$$

$$\begin{aligned}
&\leq (1-p)^{k-1} + \sum_{e\in A_{T'}} (1-p)^{k-1-|V_{T'}(e)|} \\
&\leq (1-p)^{k-1} + (1-(1-p)^{k-2})(1-p)/p \\
&= (1-(1-p)^{k-1})(1-p)/p.
\end{aligned}$$

One then concludes from (4.7) and (4.8).

■

In the second result of this section, we bound the change of the objective function of an optimal PMST occuring when one point is dropped from $x^{(n)}$.

Lemma 4.3 *Let $(x_i)_i$ be an arbitrary sequence of points in $\mathbf{R}^2$ and p be the probability of existence of each point; let $x_i^{(n)} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, then we have*

$$El_p^{pmstp}(x_i^{(n)}) \leq El_p^{pmstp}(x^{(n)}) + \sum_{x_j \in N_{T^*}(i)} \beta_{T^*}(x_i, x_j) d(x_i, x_j), \tag{4.9}$$

where $N_{T^*}(i)$ is the set of neighbors of x_i in an optimal PMSTP tree T^* through $x^{(n)}$.

Proof:
Let T^* be an optimal PMSTP tree through $x^{(n)}$ and let y be an element of $N_{T^*}(i)$ such that $d(x_i, y)$ is minimal. Let $N'_{T^*}(i)$ denote $N_{T^*}(i) \setminus \{y\}$. We get a connected graph spanning $x_i^{(n)}$ by taking the edges of T^*, deleting all the edges incident to x_i, and adding the edges which join y to the other neighbors of x_i. Let T_i be this connected graph. It has an expected length El_{T_i} such that

$$El_p^{pmstp}(x_i^{(n)}) \leq El_{T_i} = \sum_{e \in A_{T_i}} \beta_{T_i}(e) d(e). \tag{4.10}$$

Let us compare El_{T_i} with $El_p^{pmstp}(x^{(n)}) = \sum_{e\in A_{T^*}} \beta_{T^*}(e) d(e)$. We have

$$El_{T_i} = \sum_{e \in A_{T_i} \cap A_{T^*}} \beta_{T_i}(e) d(e) + \sum_{e \in A_{T_i} \setminus (A_{T_i} \cap A_{T^*})} \beta_{T_i}(e) d(e), \tag{4.11}$$

and

$$El_p^{pmstp}(x^{(n)}) = \sum_{e \in A_{T_i} \cap A_{T^*}} \beta_{T^*}(e) d(e) + \sum_{e \in A_{T^*} \setminus (A_{T_i} \cap A_{T^*})} \beta_{T^*}(e) d(e). \tag{4.12}$$

Now for all $e \in A_{T_i} \cap A_{T^*}$, choose $V_{T^*}(e)$ and $V_{T_i}(e)$ to be the subsets that contain y, so that $|V_{T_i}(e)| = |V_{T^*}(e)| - 1$. We then have

$$\begin{aligned}
\beta_{T_i}(e) &= (1-(1-p)^{|V_{T_i}(e)|})(1-(1-p)^{n-1-|V_{T_i}(e)|}) \\
&= (1-(1-p)^{|V_{T^*}(e)|-1})(1-(1-p)^{n-1-|V_{T^*}(e)|+1}) \\
&\leq (1-(1-p)^{|V_{T^*}(e)|})(1-(1-p)^{n-|V_{T^*}(e)|}) \\
&= \beta_{T^*}(e),
\end{aligned} \tag{4.13}$$

which, together with (4.10), (4.11) and (4.12), implies that

$$\begin{aligned} El_p^{pmstp}(x_i^{(n)}) &\leq El_p^{pmstp}(x^{(n)}) + \sum_{x_j \in N'_{T^*}(i)} \beta_{T_i}(y, x_j) d(y, x_j) \\ &\quad - \sum_{x_j \in N'_{T^*}(i)} \beta_{T^*}(x_i, x_j) d(x_i, x_j) - \beta_{T^*}(x_i, y) d(x_i, y). \quad (4.14) \end{aligned}$$

By triangle inequality we have $d(y, x_j) \leq d(y, x_i) + d(x_i, x_j)$ and by definition of y we have, for all $x_j \in N_{T^*}(i)$, $d(y, x_i) \leq d(x_i, x_j)$. We then have for all $x_j \in N_{T^*}(i)$

$$d(y, x_j) \leq 2d(x_i, x_j). \quad (4.15)$$

Also for all $x_j \in N'_{T^*}(i)$, choose $V_{T^*}(x_i, x_j)$ and $V_{T_i}(y, x_j)$ to be the subsets that contain y, so that $|V_{T_i}(y, x_j)| = |V_{T^*}(x_i, x_j)| - 1$. We then have

$$\begin{aligned} \beta_{T_i}(y, x_j) &= (1 - (1 - p)^{|V_{T_i}(y, x_j)|})(1 - (1 - p)^{n-1-|V_{T_i}(y, x_j)|}) \\ &= (1 - (1 - p)^{|V_{T^*}(x_i, x_j)|-1})(1 - (1 - p)^{n-1-|V_{T^*}(x_i, x_j)|+1}) \\ &\leq (1 - (1 - p)^{|V_{T^*}(x_i, x_j)|})(1 - (1 - p)^{n-|V_{T^*}(x_i, x_j)|}) \\ &= \beta_{T^*}(x_i, x_j). \quad (4.16) \end{aligned}$$

From (4.14), (4.15), and (4.16) we finally get

$$El_p^{pmstp}(x_i^{(n)}) \leq El_p^{pmstp}(x^{(n)}) + \sum_{x_j \in N'_{T^*}(i)} \beta_{T^*}(x_i, x_j) d(x_i, x_j), \quad (4.17)$$

which implies (4.9).

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4.3 Bounds For Finite Size Problems

In this section we consider sequences of points in $[0, 1]^2$ and derive upper and lower bounds on the expected length of an optimal PMSTP tree through the first n points of the sequence.

Lemma 4.4 *Let $(x_i)_i$ be an arbitrary sequence of points in $[0, 1]^2$ and p be the probability of existence of each point, then:*

$$El_p^{pmstp}(x^{(n)}) \leq \begin{cases} (1 - (1 - p)^{\lceil n/2 \rceil})^2 (\sqrt{n - 2} + 7/4) & \text{if } n \geq 3, \\ (1 - (1 - p)^{\lceil n/2 \rceil})^2 (n/4 + 2) & \text{otherwise.} \end{cases} \quad (4.18)$$

Proof:
From Lemma 4.1 we have

$$El_p^{pmstp}(x^{(n)}) \leq (1 - (1 - p)^{\lceil n/2 \rceil})^2 L_{mstp}(x^{(n)}). \quad (4.19)$$

Let $L_{mstp}^{(1)}(x^{(n)})$ be the length of an optimal MSTP tree through $x^{(n)}$ when the distance between points is the l_1 metric (i.e., rectangular metric). We obviously have

$$L_{mstp}(x^{(n)}) \leq L_{mstp}^{(1)}(x^{(n)}). \tag{4.20}$$

Now the important fact is that $L_{mstp}^{(1)}$ is a monotone functional. Suppose the square $[0,1]^2$ is described by $0 \leq h \leq 1$ (horizontal axis) and $0 \leq v \leq 1$ (vertical axis). Let the n points of $x^{(n)}$ have co-ordinates $(h_1, v_1), \ldots, (h_n, v_n)$. Divide the square into $2q$ rows of equal width (q being a positive integer to be chosen later); the square is then composed of $2q+1$ horizontal lines and 2 vertical lines. The intersections of the horizontal lines with the vertical lines give $4q+2$ points that we add to the set $x^{(n)}$. We construct a tree spanning $x^{(n)}$ and $3q+2$ of these intersection points consisting of (i) the $q+1$ horizontal lines $0 \leq h \leq 1, v = 0, 1/q, 2/q, \ldots, 1$; (ii) the n vertical links connecting each point of $x^{(n)}$ to the nearest such line, and (iii) the vertical line $h = 0, 0 \leq v \leq 1$. The length of this tree is

$$l_1 = q + 1 + \sum_{i=1}^{n} d_i + 1, \tag{4.21}$$

where d_i is the length of the vertical link from x_i to its nearest horizontal line.
We construct a similar spanning tree through $x^{(n)}$ and $3q-1$ intersection points. It consists of (i) the q horizontal lines $0 \leq h \leq 1, v = 1/2q, 3/2q, \ldots, (2q-1)/2q$; (ii) the n vertical links connecting each point of $x^{(n)}$ to the nearest such line, and (iii) the vertical line $h = 0, 1/2q \leq v \leq (2q-1)/2q$. The length of this tree is

$$l_2 = q + \sum_{i=1}^{n} d_i' + 1 - 1/q, \tag{4.22}$$

where d_i' is the length of the vertical link from x_i to its nearest horizontal line.
From the definition of $L_{mstp}^{(1)}$, and from its monotony, we have $L_{mstp}^{(1)}(x^{(n)}) \leq l_1$, and $L_{mstp}^{(1)}(x^{(n)}) \leq l_2$. Hence we have

$$L_{mstp}^{(1)}(x^{(n)}) \leq (l_1 + l_2)/2. \tag{4.23}$$

Since $d_i + d_i' = 1/2q$ for all $i \in [1..n]$, we get from (4.20), (4.21), (4.22), and (4.23)

$$L_{mstp}(x^{(n)}) \leq (2q + 1 + n/2q + 2 - 1/q)/2 = (2q + (n-2)/2q + 3)/2. \tag{4.24}$$

Finally, by chosing the best integer q in (4.24), we get, together with (4.19), the bounds of Lemma 4.4.

∎

If we make additional assumptions on the position of the points we can also derive lower bound as shown in the next lemma.

Lemma 4.5 *Let $(X_i)_i$ be a sequence of points independently and uniformly distributed over $[0,1]^2$ and p be the probability of existence of each point, then:*

$$\mathbf{E}[El_p^{pmstp}(X^{(n)})] \geq \frac{p\sqrt{n}}{2}(1-1/n)(1-(1-p)^{n-1}). \tag{4.25}$$

Proof:

From Lemma 4.1 we have

$$\mathbf{E}[El_p^{pmstp}(X^{(n)})] \geq p(1-(1-p)^{n-1})\mathbf{E}[L_{mstp}(X^{(n)})]. \tag{4.26}$$

Now let us show that

$$\mathbf{E}[L_{mstp}(X^{(n)})] \geq (n-1)/(2\sqrt{n}). \tag{4.27}$$

Let D_i denote the distance of X_i from the nearest of $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$. It is then obvious that

$$L_{mstp}(X^{(n)}) \geq \sum_{i=1}^{n-1} D_i,$$

which implies that

$$\mathbf{E}[L_{mstp}(X^{(n)})] \geq \sum_{i=1}^{n-1} E_i[E_i^c[D_i]], \tag{4.28}$$

where E_i is the expectation over X_i, and E_i^c is the conditional expectation over $\{X_1, \ldots, X_n\}$ given X_i.

Let $\mathcal{C}_r$ denote a circle of radius r centered at X_i and V_r be the surface of $\mathcal{C}_r \cap [0,1]^2$. We then have

$$\begin{aligned} E_i^c[D_i] &= \int_0^\infty \mathbf{P}(D_i > r | X_i) dr, \\ &= \int_0^\infty (1-V_r)^{n-1} dr. \end{aligned} \tag{4.29}$$

Since $(1-z)^{n-1}$ is a non-increasing non-negative function of z for $0 \leq z \leq 1$, and since $V_r \leq \min\{\pi r^2, 1\}$, equation (4.29) leads to:

$$\begin{aligned} E_i^c[D_i] &\geq \int_0^{1/\sqrt{\pi}} (1-\pi y^2)^{n-1} dy = \frac{1}{2\sqrt{\pi}} \int_0^1 x^{-1/2}(1-x)^{n-1} dx \\ &= \frac{1}{2\sqrt{\pi}} B(1/2, n) = \Gamma(n)/2\Gamma(n+1/2). \end{aligned} \tag{4.30}$$

Let $a_n = \Gamma(n) n^{1/2} / \Gamma(n+1/2)$. From Stirling formula we have $\lim_{n\to\infty} a_n = 1$, and, since $a_n/a_{n+1} = (1+1/2n)(1+1/n)^{-1/2} \geq 1$, we have from (4.30)

$$E_i^c[D_i] \geq n^{-1/2}/2. \tag{4.31}$$

Together with (4.28), this shows the validity of (4.27). The final result then follows from (4.26) and (4.27).

∎

4.4 Asymptotic Analysis

The objective of this analysis is to obtain the limiting behavior of $\mathbf{E}[El_p^{pmstp}(X^{(n)})]$.

Theorem 4.1 *Let $(X_i)_i$ be an infinite sequence of points independently and uniformly distributed over $[0,1]^2$ and p be the probability of existence of each point. Then there exists a positive constant $d(p)$ such that:*

$$\lim_{n\to\infty} \frac{\mathbf{E}[El_p^{pmstp}(X^{(n)})]}{\sqrt{n}} = d(p) \tag{4.32}$$

Note:
The proof of this result cannot be based on the asymptotics of subadditive functionals as derived in [13], because the PMSTP functional is not *monotone*. Also it cannot be based directly on the method developed by [1] for the TSP, since the PMSTP functional, in addition to being not monotone, does not seem to verify several properties on which the method of [1] is directly based. For example it would require that, for any finite collection of squares Q_j, $1 \le j \le s$, we have

$$\sum_{j=1}^{s} El_p^{pmstp}(x^{(n)} \cap Q_j) \le El_p^{pmstp}(x^{(n)} \cap (\cup_{j=1}^{s} Q_j)) + O(1), \tag{4.33}$$

and we have not been able to show it for the PMSTP. Nevertheless, having been inspired by ideas contained in [1] and [15], we will prove this theorem with the help of the following two lemmas, consequences of results stated in our previous subsections.

Lemma 4.6 *Let $(x_i)_i$ be an arbitrary sequence of points in $\mathbf{R}^2$ and p be the probability of existence of each point; let $(A_i)_{1\le i\le s}$ be a partition of a given bounded set A, then we have:*

$$\begin{aligned} El_p^{pmstp}(x^{(n)} \cap A) \quad &\le \quad \sum_{i=1}^{s} El_p^{pmstp}(x^{(n)} \cap A_i) + \sum_{i=1}^{s-1} \delta(A_i \cup A_{i+1}) \\ &\quad + ((1-p)/p) \sum_{i=1}^{s} \delta(A_i), \end{aligned} \tag{4.34}$$

where $\delta(S)$ denotes the diameter of a set S.

Proof:
The proof is similar to the demonstration of the subadditivity property of the PTSP functional developed in Section 3.2. Consider the following tree through the sequence $x^{(n)} \cap A$: first construct the optimal PMSTP tree through $x^{(n)} \cap A_i$ for $1 \le i \le s$. Then, in each subset A_i where $x^{(n)} \cap A_i$ is not empty, choose one leaf as a representative, say y_i, and consider it as *always* present; finally construct a tree through the set S of all representatives (at most s points) by simply connecting y_i

to y_{i+1} for $1 \leq i \leq s-1$. The combination of the small PMSTP subtrees together with the tree connecting the representatives gives a spanning tree through $x^{(n)} \cap A$.

By the fact that the representatives are always present this spanning tree has an expected length (in the PMSTP sense) bounded from above by:

$$\sum_{i=1}^{s} \psi_p^{pmstp}(x^{(n)} \cap A_i) + \sum_{i=1}^{s-1} \delta(A_i \cup A_{i+1}), \tag{4.35}$$

where $\psi_p^{pmstp}(x^{(n)} \cap A_i)$ denotes the new expected length of the PMSTP tree (initially constructed in A_i).

The expected length of this tree decreases if one turns back each representative into a normal point; thus we obtain a tree through $x^{(n)} \cap A$ of expected length bounded from above by (4.35) and from below by $El_p^{pmstp}(x^{(n)} \cap A)$. Hence we have:

$$El_p^{pmstp}(x^{(n)} \cap A) \leq \sum_{i=1}^{s} \psi_p^{pmstp}(x^{(n)} \cap A_i) + \sum_{i=1}^{s-1} \delta(A_i \cup A_{i+1}). \tag{4.36}$$

Now, from Lemma 4.2 one can deduce that, for all $1 \leq i \leq s$, we have:

$$\psi_p^{pmstp}(x^{(n)} \cap A_i) \leq El_p^{ptsp}(x^{(n)} \cap A_i) + ((1-p)/p)\delta(A_i). \tag{4.37}$$

Finally from (4.36) and (4.37) we get the desired result.

■

Remark:
When A is the set $[0,t]^2$, and when $(A_i)_{1 \leq i \leq m^2}$ is a partition of the square $[0,t]^2$ into squares with edges parallel to the axle and of length t/m (labeled from top left to bottom in a serpentine way), we have

$$\sum_{i=1}^{m^2-1} \delta(A_i \cup A_{i+1}) + ((1-p)/p) \sum_{i=1}^{m^2} \delta(A_i) \leq m^2(\sqrt{5}t/m + ((1-p)/p)\sqrt{2}t/m),$$

which implies that there exists a positive constant $B(p) = \sqrt{5} + ((1-p)/p)\sqrt{2}$ such that

$$El_p^{pmstp}(x^{(n)} \cap [0,t]^2) \leq \sum_{i=1}^{m^2} El_p^{pmstp}(x^{(n)} \cap A_i) + B(p)tm. \tag{4.38}$$

Equation (4.38) simply says that the PMSTP functional is subadditive in the sense of [13] (see also the proof of Theorem 3.1).

Lemma 4.7 *Let $(X_i)_i$ be a sequence of points independently and uniformly distributed over $[0,1]^2$ and p be the probability of existence of each point, then, for $n \geq 1$,*

$$(n-1)^2\mathbf{E}[El_p^{pmstp}(X^{(n-1)})] \leq n^2\mathbf{E}[El_p^{pmstp}(X^{(n)})]. \tag{4.39}$$

Proof:
By summing inequality (4.9) over $1 \leq i \leq n$ we get

$$\sum_{i=1}^{n} El_p^{pmstp}(x_i^{(n)}) \leq nEl_p^{pmstp}(x^{(n)}) + \sum_{i=1}^{n} \sum_{x_j \in N_{T^*}(i)} \beta_{T^*}(x_i, x_j) d(x_i, x_j),$$

which implies that

$$n\mathbf{E}[El_p^{pmstp}(X^{(n-1)})] \leq (n+2)\mathbf{E}[El_p^{pmstp}(X^{(n)})]. \tag{4.40}$$

Multiplying both sides of (4.40) by $(n - 2 + 1/n)$, we obtain the desired result.

■

Proof of Theorem 4.1: The asymptotics of $\mathbf{E}[El_p^{pmstp}(X^{(n)})]$ can now be obtained by a technique of Poisson smoothing followed by a Tauberian argument.
Poisson smoothing:
Let π be a Poisson point process on $\mathbf{R}^2$ with constant intensity equal to 1. For any bounded Borel set A, $El_p^{pmstp}(\pi(A))$ will denote the expected length in the PMSTP sense of an optimal PMSTP tree through the finite set of points $\pi(A)$. Now let $\phi(t) = \mathbf{E}[El_p^{pmstp}(\pi([0,t]^2))]$. By taking expectation in (4.38) we can deduce that if $(A_i)_{1 \leq i \leq m^2}$ is a partition of the square $[0,t]^2$ into squares with edges parallel to the axle and of length t/m, then there exists a positive constant $B(p)$ such that

$$\phi(t) \leq m^2 \phi(t/m) + B(p)tm. \tag{4.41}$$

From (4.41) and the continuity of ϕ, let us show that there exists a constant $d(p) \geq 0$ such that

$$\phi(t) \sim d(p)t^2 \text{ as } t \to \infty. \tag{4.42}$$

Setting $t = mu$ and dividing by $(mu)^2$ in (4.41) yields

$$\phi(mu)/(mu)^2 \leq \phi(u)/u^2 + B(p)/u. \tag{4.43}$$

Now let $d(p) = \liminf_{u \to \infty} \phi(u)/u^2 \geq 0$. From the continuity of ϕ and the definition of a lim inf, one can find, for any $\varepsilon > 0$, an interval $[u_0, u_1]$ such that

$$\phi(u)/u^2 + B(p)/u \leq d(p) + \varepsilon$$

for all $u \in [u_0, u_1]$. From (4.43) this implies that, for all integer m, we have for $u \in [u_0, u_1]$

$$\phi(mu)/(mu)^2 \leq d(p) + \varepsilon. \tag{4.44}$$

If we let $\mathbf{B} = \{t \in \mathbf{R} : \phi(t)/t^2 \leq d(p) + \varepsilon\}$, then from (4.44) we have

$$\cup_{m=1}^{\infty} [mu_0, mu_1] \subset \mathbf{B}. \tag{4.45}$$

Moreover, by choosing $m_0 = u_0/(u_1 - u_0)$, the intervals $[mu_0, mu_1]$ are overlapping for $m \geq m_0$, and so (4.45) implies that

$$[m_0 u_0, \infty[\subset \mathbf{B}, \tag{4.46}$$

which implies that $\limsup_{u\to\infty} \phi(u)/u^2 \le d(p)+\varepsilon$, and this terminates the proof of (4.42).

Tauberian argument:

By the definition of $\phi(t)$ and by scaling property we have

$$\phi(t) = t\sum_{n=0}^{\infty} \varphi_n \frac{e^{-t^2}t^{2n}}{n!}, \tag{4.47}$$

where $\varphi_n = \mathbf{E}[El_p^{pmstp}(X^{(n)})]$. Setting $t=\sqrt{u}$ in (4.47) and using (4.42), we have

$$\lim_{u\to\infty} \frac{\phi(\sqrt{u})}{u} = \lim_{u\to\infty} u^{-1/2}\sum_{n=0}^{\infty} \varphi_n \frac{e^{-u}u^n}{n!} = d(p). \tag{4.48}$$

We are now going to use a classical Tauberian theorem due to Schmidt [12] (see [4]):

Theorem (Schmidt): *If we have*

$$\lim_{x\to\infty} \sum_{n=0}^{\infty} a_n \frac{e^{-x}x^n}{n!} = s, \tag{4.49}$$

then

$$\lim_{n\to\infty} a_n = s$$

if and only if

$$\lim_{\varepsilon\to 0^+} \liminf_{n\to\infty} \min_{n\le m\le n+\varepsilon\sqrt{n}} \{a_m - a_n\} \ge 0. \tag{4.50}$$

In order to use this theorem, first let us show that

$$\lim_{u\to\infty} \sum_{n=0}^{\infty} \frac{\varphi_n}{\sqrt{n}} \frac{e^{-u}u^n}{n!} = d(p). \tag{4.51}$$

Let $\varphi(u) = \sum_{n=0}^{\infty} (\varphi_n/\sqrt{n})e^{-u}u^n/n!$. From Lemma 4.4 we know that there exists a constant, say C, such that

$$\varphi_n \le C\sqrt{n}. \tag{4.52}$$

Hence, we have for $0<\varepsilon<1$

$$\begin{aligned}\varphi(u) &= \sum_{n=0}^{\lfloor u(1-\varepsilon)\rfloor} \frac{\varphi_n}{\sqrt{n}}\frac{e^{-u}u^n}{n!} + \sum_{n=\lfloor u(1-\varepsilon)\rfloor+1}^{\infty} \frac{\varphi_n}{\sqrt{n}}\frac{e^{-u}u^n}{n!} \\ &\le C\sum_{n=0}^{\lfloor u(1-\varepsilon)\rfloor} \frac{e^{-u}u^n}{n!} + (\lfloor u(1-\varepsilon)\rfloor)^{-1/2}\sum_{n=0}^{\infty} \varphi_n \frac{e^{-u}u^n}{n!}. \end{aligned} \tag{4.53}$$

From the behavior of the probability in the tail of a Poisson distribution the first term in (4.53) goes to zero when u goes to infinity, so that (4.48) and (4.53) give

$$\limsup_{u\to\infty} \varphi(u) \le (1-\varepsilon)^{-1/2} d(p). \tag{4.54}$$

Also we have

$$
\begin{aligned}
\varphi(u) &\geq \sum_{n=0}^{\lfloor u(1+\varepsilon)\rfloor} \frac{\varphi_n}{\sqrt{n}} \frac{e^{-u}u^n}{n!} \geq (\lfloor u(1+\varepsilon)\rfloor)^{-1/2} \sum_{n=0}^{\lfloor u(1+\varepsilon)\rfloor} \varphi_n \frac{e^{-u}u^n}{n!} \\
&= (\lfloor u(1+\varepsilon)\rfloor)^{-1/2} \left[\sum_{n=0}^{\infty} \varphi_n \frac{e^{-u}u^n}{n!} - \sum_{n=\lfloor u(1+\varepsilon)\rfloor+1}^{\infty} \varphi_n \frac{e^{-u}u^n}{n!} \right] \\
&\geq (\lfloor u(1+\varepsilon)\rfloor)^{-1/2} \left[\sum_{n=0}^{\infty} \varphi_n \frac{e^{-u}u^n}{n!} - Cu \sum_{n=\lfloor u(1+\varepsilon)\rfloor}^{\infty} \frac{e^{-u}u^n}{n!} \right], \quad (4.55)
\end{aligned}
$$

where we have used in the last inequality the fact that $\varphi_n \leq C\sqrt{n} \leq Cn$. Here again, from the behavior of the probability in the tail of a Poisson distribution the second term in (4.55) goes to zero when u goes to infinity, so that (4.48) and (4.55) give

$$\liminf_{u\to\infty} \varphi(u) \geq (1+\varepsilon)^{-1/2} d(p). \tag{4.56}$$

Equations (4.54) and (4.56) finally prove (4.51).
In order to conclude it remains to prove that $\varphi_n/\sqrt{n}$ verifies the condition (4.50). From Lemma 4.7 we have $m^2\varphi_m \geq n^2\varphi_n$ for all $m \geq n$. Hence we have

$$\varphi_m/\sqrt{m} - \varphi_n/\sqrt{n} \geq \varphi_n/\sqrt{n}((n/m)^{5/2} - 1). \tag{4.57}$$

Also from Lemma 4.5 we know that there exists a positive constant C' such that $\varphi_n/\sqrt{n} \geq C'$. So finally we have for $n \leq m \leq n + \varepsilon\sqrt{n}$

$$\varphi_m/\sqrt{m} - \varphi_n/\sqrt{n} \geq C'((1+\varepsilon/\sqrt{n})^{-5/2} - 1), \tag{4.58}$$

which shows the validity of (4.50) for $\varphi_n/\sqrt{n}$.

∎

The constant $d(p)$ is unknown and the only bounds available are given by the following lemma:

Lemma 4.8 *We have for all p in $[0,1]$:*

$$p/2 \leq d(p) \leq \beta_{mstp}, \tag{4.59}$$

where β_{mstp} is the "MSTP-constant".

Proof:
The lower bound follows from Lemma 4.5 and the upper bound from Lemma 4.1 and the fact that $\mathbf{E}[L_{mstp}(X^{(n)})]/\sqrt{n}$ goes to a constant β_{mstp} when n goes to infinity (see for example [15]).

∎

5 Analysis of the Re-Optimization Strategies

We are interested, in this section, in the behaviors of $O_p^{ptsp}(X^{(n)})$ and $O_p^{pmstp}(X^{(n)})$ when n goes to infinity. We have the following theorem:

Theorem 5.1 *Let $(X_i)_i$ be an infinite sequence of points independently and uniformly distributed over $[0,1]^2$ and p be the probability of existence of each point. Then we have:*

$$\lim_{n\to\infty} \frac{O_p^{ptsp}(X^{(n)})}{\sqrt{n}} = \beta_{tsp}\sqrt{p} \ (a.s.), \tag{5.60}$$

and

$$\lim_{n\to\infty} \frac{O_p^{pmstp}(X^{(n)})}{\sqrt{n}} = \beta_{mstp}\sqrt{p} \ (a.s.), \tag{5.61}$$

where β_{tsp} and β_{mstp} are respectively the "TSP-constant" and the "MSTP-constant".

Proof:
Let us first prove (5.60). We will suppose that the sequences $(X_i)_i$ and $(Y_i)_i$ (see Section 2) are defined on their canonical probability space, here denoted respectively $(\Omega_1, \mathcal{A}_1, \mathbf{P}_1)$ and $(\Omega_2, \mathcal{A}_2, \mathbf{P}_2)$. If $L_{tsp}(A)$ denotes the length of an optimal TSP tour through the finite set of points A, let $(L_n)_n$ be a sequence of random variables defined on $(\Omega_1, \mathcal{A}_1, \mathbf{P}_1)$ by:

$$\forall n \in \mathbf{N}, L_n = L_{tsp}(X_j; 1 \leq j \leq n).$$

Let $(I_n)_n$ and $(M_n)_n$ be sequences of random variables defined on $(\Omega_2, \mathcal{A}_2, \mathbf{P}_2)$ by:

$$\forall n \in \mathbf{N}, \left\{ \begin{array}{lcl} I_n & = & \{j : 1 \leq j \leq n \text{ and } Y_j = 1\}, \\ M_n & = & \sum_{j=1}^n \mathbf{1}_{\{Y_j = 1\}}. \end{array} \right.$$

By definition, the objective function of the re-optimization strategy is

$$O_p^{ptsp}(X^{(n)}) = \int_{\Omega_2} L_{tsp}(X_j; j \in I_n) d\mathbf{P}_2. \tag{5.62}$$

From [14] we know that there exists a positive constant β_{tsp} such that

$$\forall \varepsilon > 0, \sum_n \mathbf{P}_1(|L_n/\sqrt{n} - \beta_{tsp}| > \varepsilon) < +\infty, \tag{5.63}$$

(this is a complete convergence and it implies that $\lim_{n\to\infty} L_n/\sqrt{n} = \beta_{tsp}$ ($\mathbf{P}_1$-a.s.)). Also from the strong law of large numbers we have

$$\lim_{n\to\infty} M_n/n = p \ (\mathbf{P}_2\text{-a.s.}). \tag{5.64}$$

Now the important fact is that for any fixed $\omega_2 \in \Omega_2$ and for any n, the random variables $L_{tsp}(X_j; 1 \leq j \leq M_n(\omega_2))$ and $L_{tsp}(X_j; j \in I_n(\omega_2))$ have the same distribution. From (5.63) and (5.64) we then have

$$\lim_{n\to\infty} L_{tsp}(X_j; j \in I_n)/\sqrt{n} = \beta_{tsp}\sqrt{p} \ (\mathbf{P}_1 \otimes \mathbf{P}_2\text{-a.s.}). \tag{5.65}$$

Finally, from Lemma 3.1 applied with $p = 1$, we know that $L_{tsp}(X_j; j \in I_n)/\sqrt{n}$ is uniformly bounded by a constant. It is then easy to conclude from (5.65) and Lebesgue's dominated convergence theorem.
In order to prove (5.61), the argument is identical if one shows that there is complete convergence for the MSTP functional (i.e., if one can get a result similar to (5.63)). In order to do that we will use the following result , proved in [15],

$$\lim_{n\to\infty} \frac{\mathbf{E}[L_{mstp}(X^{(n)})]}{\sqrt{n}} = \beta_{mstp}\sqrt{p}, \tag{5.66}$$

and a martingale inequality argument due to [11]. Let Σ_i be the σ-algebra generated by $(X_k)_{k\leq i}$ and let $h_i = L_{mstp}(X^{(n)}) - L_{mstp}(X_i^{(n)})$, where $X_i^{(n)} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$. Then

$$d_i \stackrel{\text{def}}{=} \mathbf{E}[L_{mstp}(X^{(n)})|\Sigma_i] - \mathbf{E}[L_{mstp}(X^{(n)})|\Sigma_{i-1}] = \mathbf{E}[h_i|\Sigma_i] - \mathbf{E}[h_i|\Sigma_{i-1}]. \tag{5.67}$$

Moreover we have $L_{mstp}(X^{(n)}) - \mathbf{E}[L_{mstp}(X^{(n)})] = \sum_{i=1}^n d_i$. Now, since $(d_i)_{i\leq n}$ is a martingale difference sequence, we can use the following martingale inequality (due to Azuma, see [11])

$$\mathbf{P}(|\sum_{i=1}^n d_i| > t) \leq 2\exp(-t^2/(2\sum_{i=1}^n \|d_i\|_\infty^2)), \tag{5.68}$$

in order to get bounds on

$$\mathbf{P}(|L_{mstp}(X^{(n)}) - \mathbf{E}[L_{mstp}(X^{(n)})]| > t).$$

It remains to control the numbers $\|d_i\|_\infty$. In order to do that, let us prove the following lemma:

Lemma 5.1 *There exists a numerical constant K such that one has:*

$$\begin{aligned} |h_i| &\leq K \text{ for } i \leq n, \text{ and} && (5.69)\\ \mathbf{E}[h_i|\Sigma_i] &\leq K/\sqrt{n-i} \text{ for } i \leq n-1. && (5.70) \end{aligned}$$

Proof of Lemma 5.1:
In order to construct a tree through $X^{(n)}$, one can complete a tree through $X_i^{(n)}$ by adding an edge from X_i to one of the point of $X_i^{(n)}$. We then have $L_{mstp}(X^{(n)}) \leq L_{mstp}(X_i^{(n)}) + \sqrt{2}$ and this takes care of the first inequalities (take $K \geq \sqrt{2}$). Now let l_i denote the distance of X_i from the nearest of $X_{i+1}, \ldots, X_n$. We then have

$$L_{mstp}(X^{(n)}) \leq L_{mstp}(X_i^{(n)}) + l_i. \tag{5.71}$$

Also we have

$$\mathbf{E}[l_i|\Sigma_i] = E_i^c[l_i], \tag{5.72}$$

where E_i^c is the conditional expectation given X_i.
Let $\mathcal{C}_r$ denote a disc of radius r centered at X_i and V_r be the volume of $\mathcal{C}_r \cap [0,1]^2$. We then have

$$\begin{aligned} E_i^c[l_i] &= \int_0^\infty \mathbf{P}(l_i > r|X_i)dr, \\ &= \int_0^\infty (1-V_r)^{n-i}dr. \end{aligned} \tag{5.73}$$

Since $(1-z)^{n-i}$ is a non-increasing non-negative function of z for $0 \le z \le 1$, and since $V_r \ge \min\{r^2/2, 1\}$, (5.73) leads to:

$$\begin{aligned} E_i^c[l_i] &\le \int_0^{\sqrt{2}} (1-r^2/2)^{n-i}dr \\ &= \Gamma(1/2)\Gamma(n-i+1)/\sqrt{2}\Gamma(n-i+3/2) \\ &= \sqrt{\pi/2}\Gamma(n-i+1)/\Gamma(n-i+3/2). \end{aligned} \tag{5.74}$$

Let $a_n = \Gamma(n-i+1)(n-i)^{1/2}/\Gamma(n-i+3/2)$. From Stirling formula we have $\lim_{n\to\infty} a_n = 1$, and, since $a_n/a_{n+1} = (1+1/2(n-i+1))(1-1/(n-i+1))^{1/2} \le 1$, we have from (5.74)

$$E_i^c[l_i] \le \sqrt{\pi/2}/(n-i)^{1/2} \le \sqrt{2}/(n-i)^{1/2}. \tag{5.75}$$

Lemma 5.1 is thus valid with $K = \sqrt{2}$.

∎

End of proof of (5.61):
From (5.67) we have

$$\|d_i\|_\infty \le \|\mathbf{E}[h_i|\Sigma_i]\|_\infty + \|\mathbf{E}[h_i|\Sigma_{i-1}]\|_\infty \le 2\|\mathbf{E}[h_i|\Sigma_i]\|_\infty. \tag{5.76}$$

Equation (5.76) together with Lemma 5.1 implies that $\|d_i\|_\infty \le 2K/\sqrt{n-i}$ for $i \le n-1$ and $\|d_n\|_\infty \le 2K$. Replacing these bounds into equation (5.68) we finally get

$$\mathbf{P}(|L_{mstp}(X^{(n)}) - \mathbf{E}[L_{mstp}(X^{(n)})]| > \varepsilon\sqrt{n}) \le 2\exp(-\varepsilon^2 n/(K^* \ln n)), \tag{5.77}$$

where K^* is a constant. Finally the complete convergence of the MSTP follows from (5.77), (5.66) and a "2ε" argument.

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6 Concluding remarks

In addition to the importance of the asymptotic results as described in the introduction, let us mention that Theorems 3.1 and 5.1 provide interesting practical by-products: $(c(p) - \beta_{tsp}\sqrt{p})\sqrt{n}$ (respectively $(d(p) - \beta_{mstp}\sqrt{p})\sqrt{n}$) represents an approximation of the penalty one has to pay when n potential customers have to be served and when the route (respectively tree) is not re-optimized for each instance of the problem. This estimate of the penalty is asymptotically exact with probability one for the PTSP, and in expectation for the PMSTP.

The results presented in this paper can be generalized in several directions. First, all our asymptotic results remain valid if the points are independently and uniformly distributed over $[0,t]^2$, the constants being simply multiplied by t. This remains true, for Theorem 5.1, if the points are distributed in a bounded support of Lebesgue measure t^2. Also Theorem 5.1 remains true for a non-uniform distribution of points, and can be strengthened to complete convergence.

However, some generalizations do not seem to be easy, and here is a list of the most important open problems related to the PTSP and the PMSTP:

1. the almost sure convergence of the PMSTP, and its complete convergence,
2. the complete convergence of the PTSP,
3. the non uniform extension for the PTSP and the PMSTP.

Finally let us also mention the problem of rates of convergence for the previous limit theorems. Some preliminary results have been obtained for the traveling salesman problem, the minimum spanning tree problem, and the minimum matching problem and will be report in a subsequent paper. For the probabilistic version of these problems the analysis seems much more difficult.

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