OPERATIONS RESEARCH
DOI 10.1287/opre.1070.0450ec
pp. ec1–ec6

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Electronic Companion—"Generalized Online Routing: New Competitive Ratios, Resource Augmentation and Asymptotic Analyses" by Patrick Jaillet and Michael R. Wagner, *Operations Research*, DOI 10.1287/opre.1070.0450.

Generalized Online Routing: New Competitive Ratios, Resource Augmentation and Asymptotic Analyses Online Appendix

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Subject classifications: Online Optimization, Transportation, Analysis of Algorithms.
Area of review: Transportation
History: Received May 2006; revision received November 2006; revision received February 2007; accepted April 2007.

Appendix

Proof of Theorem 1. Recall that r_n is the time of the last request and $l_n^* = \arg\max_{l_n^j \mid 1 \le j \le k(n)} d(o, l_n^j)$. We show that in each of the Cases (1), (2a) and (2b), PAH-G is $(1 + (2\rho - 1)/\gamma)$-competitive.

In Case (1) PAH-G is at the origin at time r_n. It starts traversing a ρ-approximate set of tours that serve all the unserved requests. Since the online server has speed γ, the time needed by PAH-G is at most $r_n + \rho Z^{r=0}(n,Q)/\gamma \le (1 + \rho/\gamma) Z^*(n,Q)$.

Considering Case (2a), we have that $d(o, l_n^*) > d(o,p)$. Then PAH-G goes back to the origin, where it will arrive before time $r_n + \frac{d(o,l_n^*)}{\gamma}$. After this, PAH-G computes and follows a ρ-approximate set of tours through all the unserved requests. Therefore, the online cost is at most $r_n + \frac{d(o,l_n^*)}{\gamma} + \rho Z^{r=0}(n,Q)/\gamma$. Noticing that $r_n + d(o,l_n^*) \le Z^*(n,Q)$ and $2d(o,l_n^*) \le Z^{r=0}(n,Q)$, we have that the online cost is at most

$$\begin{aligned} r_n + \frac{d(o,l_n^*)}{\gamma} + \rho \frac{Z^{r=0}(n,Q)}{\gamma} &\le Z^*(n,Q) + (\frac{1}{\gamma} - 1) d(o,l_n^*) + \frac{\rho}{\gamma} Z^*(n,Q) \\ &\le \left(1 + \left(\frac{2\rho - \gamma + 1}{2\gamma}\right)\right) Z^*(n,Q) \end{aligned}$$

Finally, we consider Case (2b), where $d(o,l_n^*) \le d(o,p)$. Suppose PAH-G is following a route $\mathcal{R}$ that had been computed the last time step (1) of PAH-G had been invoked. $\mathcal{R}$ will also denote the actual distance of the route; we have that $\mathcal{R} \le \rho Z^{r=0}(n,Q) \le \rho Z^*(n,Q)$. Let $\mathcal{S}$ be the set of requests that have been temporarily ignored (from step (2b) of algorithm PAH-G) since the last time PAH-G invoked step (1). Let l_f be the first location of the first request in $\mathcal{S}$ visited by the offline algorithm, and let r_f be the time at which request f was released. Let $\mathcal{P}_{\mathcal{S}}^*$ be the fastest route that starts at l_f, visits all cities in $\mathcal{S}$ and ends at the origin, respecting precedence and capacity constraints. Clearly, $Z^*(n,Q) \ge r_f + \mathcal{P}_{\mathcal{S}}^*$ and $Z^*(n,Q) \ge d(o,l_f) + \mathcal{P}_{\mathcal{S}}^*$.

At time r_f, the time that PAH-G still has left to complete route $\mathcal{R}$ is at most $(\mathcal{R}-d(o,l_f))/\gamma$, since $d(o,p(r_f)) \geq d(o,l_f^*) \geq d(o,l_f)$ implies that PAH-G has traveled on the route $\mathcal{R}$ a distance not less than $d(o,l_f)$. Therefore, the server will complete the route $\mathcal{R}$ before time $r_f + (\mathcal{R}-d(o,l_f))/\gamma$. After that it will follow a ρ-approximate set of tours that covers the set $\mathcal{S}$ of yet unserved requests; let $\mathcal{T}_\mathcal{S}$ denote the cost of the *optimal* set of tours. Hence, the total time to completion will be at most $r_f + (\mathcal{R}-d(o,l_f))/\gamma + \rho\mathcal{T}_\mathcal{S}/\gamma$. Since $\mathcal{T}_\mathcal{S} \leq d(0,l_f) + \mathcal{P}^*_\mathcal{S}$, we have that the online cost is at most

$$\begin{aligned} r_f + \frac{\mathcal{R}-d(o,l_f)}{\gamma} + \frac{\rho}{\gamma}d(0,l_f) + \frac{\rho}{\gamma}\mathcal{P}^*_\mathcal{S} &= (r_f + \mathcal{P}^*_\mathcal{S}) + \frac{1}{\gamma}\mathcal{R} + \left(\frac{\rho-1}{\gamma}\right)d(0,l_f) + \left(\frac{\rho}{\gamma}-1\right)\mathcal{P}^*_\mathcal{S} \\ &\leq Z^*(n,Q) + \frac{\rho}{\gamma}Z^{r=0}(n,Q) + \left(\frac{\rho-1}{\gamma}\right)Z^*(n,Q) \\ &\leq \left(1+\frac{2\rho-1}{\gamma}\right)Z^*(n,Q). \end{aligned}$$

Since $\rho,\gamma \geq 1$, $\max\left\{1+\frac{\rho}{\gamma}, 1+\frac{2\rho-1}{\gamma}, 1+\left(\frac{2\rho-\gamma+1}{2\gamma}\right)\right\} = 1+\frac{2\rho-1}{\gamma}$ and the theorem is proved. □

Proof of Theorem 3. Let r_n be the time of the last request, l_n the position of this request and $p^*(t)$ the location of the farthest salesman at time t.

Case (1): All salesmen are at the origin at time r_n. Then they start implementing a ρ-approximate solution to $Z^{r=0}(n,m)$ that serves all the unserved requests. Applying Lemma 1, the time needed by PAH-m is at most

$$r_n + \frac{\rho}{\gamma}Z^{r=0}(n,m) \leq Z^*(n,1) + \frac{\rho}{\gamma}\left(Z^{r=0}(n,1)-(m-1)\beta)\right) \leq \left(1+\frac{\rho}{\gamma}(1-(m-1)\phi)\right)Z^*(n,1).$$

Case (2a): We have that $d(o,l_n) > d(o,p^*(r_n))$. All salesmen return to the origin, where they will all arrive before time $r_n + d(o,l_n)/\gamma \leq r_n + d(o,l_n)$. After this, PAH-m computes and follows a ρ-approximate solution to $Z^{r=0}(n,m)$ through all unserved requests. Therefore, the online cost is at most $r_n + d(o,l_n) + \frac{\rho}{\gamma}Z^{r=0}(n,m)$. Noticing that $r_n + d(o,l_n) \leq Z^*(n,1)$ and applying Lemma 1, we have that the online cost is at most $\left(1+\frac{\rho}{\gamma}(1-(m-1)\phi)\right)Z^*(n,1)$.

Case (2b): We have that $d(o,l_n) \leq d(o,p^*(r_n))$ and all salesmen, except p^*, return to the origin, if not yet already there. Suppose salesman p^* is following a tour $\mathcal{R}$ that had been computed the last time it was at the origin. Note that $\mathcal{R} \leq \rho Z^{r=0}(n,m)$ and $Z^{r=0}(n,m) \leq Z^*(n,m) \leq Z^*(n,1)$. Let $\mathcal{Q}$ be the set of requests temporarily ignored since the last time a Case (1) re-optimization was performed; since $l_n \in \mathcal{Q}$, $\mathcal{Q}$ is not empty. Let $\mathcal{S} \subseteq \{1,\ldots,m\}$ denote the set of salesmen that serve $\mathcal{Q}$ in the optimal offline solution. For $j \in \mathcal{S}$, let l^j be the location of the first city in $\mathcal{Q}$ served by server j in the *optimal offline solution* and let r^j be the time at which this city was released. Let $\mathcal{P}^j_\mathcal{Q}$, $j \in \mathcal{S}$, be the set of paths, the j-th path starting from l^j, that collectively visit all the cities in $\mathcal{Q}$ and end at the origin, such that the maximum path length is minimized (ties broken arbitrarily). It is easy to see that $Z^*(n,m) \geq \max_{j\in\mathcal{S}}\{\mathcal{P}^j_\mathcal{Q}\}$ since the min-max-path optimization has distinct advantages over the offline solution: (1) having the servers start at cities l^j, (2) needing to only serve the cities in $\mathcal{Q}$ and (3) ignoring release dates. If the servers start from the origin, the earliest time that server j can visit city l^j is $\max\{r^j, d(0,l^j)\}$; by extension we have that $Z^*(n,m) \geq \max_{j\in\mathcal{S}}\{r^j + \mathcal{P}^j_\mathcal{Q}\}$ and $Z^*(n,m) \geq \max_{j\in\mathcal{S}}\{d(o,l^j) + \mathcal{P}^j_\mathcal{Q}\}$.

At time r^j, the distance that salesman p^* still has to travel on the route $\mathcal{R}$ before arriving at the origin is at most $\mathcal{R} - d(o,l^j)$, since $d(o,p^*(r^j)) \geq d(o,l^j)$ implies that p^* has traveled on the route $\mathcal{R}$ a distance not less than $d(o,l^j)$. Therefore, it will arrive at the origin before time $r^j + \frac{\mathcal{R}-d(o,l^j)}{\gamma}$; note that since this is valid for any j, we can say that the salesman will arrive at the origin before time $\min_{j\in\mathcal{S}}\{r^j + \frac{\mathcal{R}-d(o,l^j)}{\gamma}\}$. Note that all other salesmen have already arrived at the origin. Next, a ρ-approximate $Z^{r=0}(n,m)$ will be implemented on $\mathcal{Q}$; let $\mathcal{T}_\mathcal{Q}$ denote the *optimal* maximum tour length. Hence, the completion time of PAH-m will be at most $\min_{j\in\mathcal{S}}\{r^j + \frac{\mathcal{R}-d(o,l^j)}{\gamma}\} + \frac{\rho}{\gamma}\mathcal{T}_{\mathcal{Q}}$. Now, note the following feasible solution for the final case (1) re-optimization: Use only the set of salesmen $\mathcal{S}$, force salesman j to first go to city l^j and then traverse path $\mathcal{P}^j_{\mathcal{Q}}$. Therefore, $\mathcal{T}_{\mathcal{Q}} \leq \max_{j\in\mathcal{S}}\{d(0,l^j)+\mathcal{P}^j_{\mathcal{Q}}\}$ and we have that the online cost is at most

$$\min_{j\in\mathcal{S}}\left\{r^j + \frac{\mathcal{R}-d(o,l^j)}{\gamma}\right\} + \rho\max_{j\in\mathcal{S}}\left\{\frac{d(0,l^j)+\mathcal{P}^j_{\mathcal{Q}}}{\gamma}\right\}.$$

Letting k be the arg max of the second term, we have that the online cost is at most

$$\begin{aligned} r^k + \frac{\mathcal{R}-d(o,l^k)+\rho\left(d(0,l^k)+\mathcal{P}^k_{\mathcal{Q}}\right)}{\gamma} &= \left(r^k+\mathcal{P}^k_{\mathcal{Q}}\right) + \frac{1}{\gamma}\mathcal{R} + \left(\frac{\rho-1}{\gamma}\right)d(o,k^k) + \left(\frac{\rho}{\gamma}-1\right)\mathcal{P}^k_{\mathcal{Q}} \\ &\leq Z^*(n,m) + \frac{\rho}{\gamma}Z^{r=0}(n,m) + \left(\frac{\rho-1}{\gamma}\right)\left(d(o,l^k)+\mathcal{P}^k_{\mathcal{Q}}\right) \\ &\leq \frac{\rho}{\gamma}\left(Z^{r=0}(n,1)-(m-1)\beta\right) + \left(\frac{\rho-1+\gamma}{\gamma}\right)Z^*(n,1) \\ &\leq \left(1+\frac{\rho}{\gamma}(1-(m-1)\phi) + \frac{\rho-1}{\gamma}\right)Z^*(n,1).\square \end{aligned}$$

Proof of Theorem 7. Let $p^*(t)$ be the position of the farthest server at time t. Let us consider the state of the algorithm at time q_n, the final disclosure date.

Case (1): All servers are at the origin at time q_n. Letting T denote the cost of the final Case (1) re-optimization, we have that

$$\begin{aligned} Z^{\text{PAH-m-dd}}(n,m) &\leq q_n + T \\ &= r_n + T - a \\ &\leq Z^*(n,m) + (T-a) \\ &= Z^*(n,m) + (1-\frac{a}{T})T \\ &\leq Z^*(n,m) + (1-\frac{a}{T})Z^*(n,m). \end{aligned}$$

Inserting the obvious bound $T \leq a + Z^{r=0}(n,m)$ proves the theorem for this case.

Case (2a): We have that $d(o,l_n) > d(o,p^*(q_n))$ and the servers return to the origin, arriving before time $q_n + d(o,l_n) = r_n + d(o,l_n) - a$. Once at the origin, the servers re-optimize; let T' denote the cost of this re-optimization. Clearly, $r_n + d(o,l_n) \leq Z^*(n,m)$. Thus, we have that

$$Z^{\text{PAH-dd}}(n,m) \leq r_n + d(o,l_n) + (T'-a) \leq Z^*(n,m) + (1-\frac{\alpha}{1+\alpha})Z^*(n,m) = (2-\frac{\alpha}{1+\alpha})Z^*(n,m).$$

Case (2b): We have that $d(o,l_n) \leq d(o,p^*(r_n))$ and all servers, except p^*, return to the origin, if not yet already there. Suppose server p^* is following a tour $\mathcal{R}$ that had been computed the last time it was at the origin. Note that $\mathcal{R} \leq Z^*(n,m)$. Let $\mathcal{Q}$ be the set of requests temporarily ignored since the last time a Case (1) re-optimization was performed; since $l_n \in \mathcal{Q}$, $\mathcal{Q}$ is not empty. Let $\mathcal{S} \subseteq \{1,\ldots,m\}$ denote the set of servers that serve $\mathcal{Q}$ in the optimal offline solution. For $j \in \mathcal{S}$, let l^j be the location of the first city in $\mathcal{Q}$ served by server j in the *optimal offline solution* and let r^j be the time at which this city was released. Let $\mathcal{P}^j_{\mathcal{Q}}$, $j \in \mathcal{S}$, be the set of paths, the j-th path starting from l^j, that collectively visit all the cities in $\mathcal{Q}$ and end at the origin, such that the maximum path length is minimized. As was argued in the proof of Theorem 3, $Z^*(n,m) \geq \max_{j\in\mathcal{S}}\{r^j + \mathcal{P}^j_{\mathcal{Q}}\}$ and $Z^*(n,m) \geq \max_{j\in\mathcal{S}}\{d(o,l^j)+\mathcal{P}^j_{\mathcal{Q}}\}$.

At time q^j, the distance that salesman p^* still has to travel on the route $\mathcal{R}$ before arriving at the origin is at most $\mathcal{R} - d(o,l^j)$, since $d(o,p^*(q^j)) \geq d(o,l^j)$ implies that p^* has traveled on the route $\mathcal{R}$ a distance not less than $d(o,l^j)$. Therefore, it will arrive at the origin before time $q^j+\mathcal{R}-d(o,l^j)$; note that since this is valid for any j, we can say that the salesman will arrive at the origin before time $\min_{j\in\mathcal{S}}\{q^j+\mathcal{R}-d(o,l^j)\}$. Note that all other salesmen have already arrived at the origin. Next, a re-optimization will be implemented on $\mathcal{Q}$; let $\mathcal{T}_\mathcal{Q}$ denote the maximum tour length. Hence, the completion time of PAH-m-dd will be at most $\min_{j\in\mathcal{S}}\{q^j+\mathcal{R}-d(o,l^j)\}+\mathcal{T}_\mathcal{Q}$. Again, $\mathcal{T}_\mathcal{Q}\le\max_{j\in\mathcal{S}}\{d(0,l^j)+\mathcal{P}^j_\mathcal{Q}\}$ and we have that the online cost is at most

$$\min_{j\in\mathcal{S}}\{q^j+\mathcal{R}-d(o,l^j)\}+\max_{j\in\mathcal{S}}\{d(0,l^j)+\mathcal{P}^j_\mathcal{Q}\}.$$

Letting k be the arg max of the second term, we have that the online cost is at most

$$q^k+\mathcal{R}-d(o,l^k)+d(0,l^k)+\mathcal{P}^k_\mathcal{Q}=\left(r^k+\mathcal{P}^k_\mathcal{Q}\right)+(\mathcal{R}-a)\le\left(2-\frac{\alpha}{1+\alpha}\right)Z^*(n,m)\ \square$$

Proof of Theorem 8. Define a metric space $\mathcal{M}$ as a graph with vertex set $V=\{1,2,\dots,n\}\cup\{o\}$ with distance function d that satisfies the following: $d(o,i)=1$ and $d(i,j)=2$ for all $i\ne j\in V\setminus\{o\}$. For simplicity, assume m divides n evenly.

At time 0, there is a request at each of the n cities in $V\setminus\{o\}$. If an online server visits the request at city i at time $t\le 2\frac{n}{m}-1-\epsilon$, for some small ϵ, then at time $t+\epsilon$, a new request is disclosed at city i. In this way, at time $2\frac{n}{m}-1$ the online servers still have to serve requests at all n cities, some of which are only disclosed and not released. If all cities were released, the online servers could finish at time $(2\frac{n}{m}-1)+(2\frac{n}{m}-1)/\gamma=(1+1/\gamma)(2\frac{n}{m}-1)$; therefore this is a lower bound for the online cost when cities have only been disclosed. Denoting $Z^A(n,m)$ as the online cost of an arbitrary online algorithm A, we have that $Z^A(n,m)\ge(1+\frac{1}{\gamma})(2\frac{n}{m}-1)$. The optimal offline servers, however, will be able to visit all cities by time $2\frac{n}{m}+a$. Therefore, by letting $k=\frac{n}{m}$ and noting that $Z^{r=0}(n,m)=2k$, we have that

$$\frac{Z^A(n,m)}{Z^*(n,m)}\ge\frac{(1+1/\gamma))(2k-1)}{2k+a}=\frac{1+1/\gamma}{1+\alpha}-\frac{1+1/\gamma}{2k+a};$$

taking k arbitrarily large proves the theorem. $\square$

Proof of Theorem 9. Define a metric space $\mathcal{M}$ as a graph with vertex set $V=\{1,2,\dots,n\}\cup\{o\}$ with distance function d that satisfies the following: $d(o,i)=1$ and $d(i,j)=2$ for all $i\ne j\in V\setminus\{o\}$. For simplicity, assume m divides n evenly.

At time 0, there is a request at each of the n cities in $V\setminus\{o\}$. If an online server visits the request at city i at time $t\le 2\frac{n}{m}-1-\epsilon$, for some small ϵ, then at time $t+\epsilon$, a new request is disclosed at city i. In this way, at time $2\frac{n}{m}-1$ the online servers still have to serve requests at all n cities, some of which are only disclosed and not released. If all cities were released, the online servers could finish at time $(2\frac{n}{m}-1)+(2\frac{n}{m}-1)/\gamma=(1+1/\gamma)(2\frac{n}{m}-1)$; therefore this is a lower bound for the online cost when cities have only been disclosed. Denoting $Z^A(n,m)$ as the online cost of an arbitrary online algorithm A, we have that $Z^A(n,m)\ge(1+\frac{1}{\gamma})(2\frac{n}{m}-1)$. The single optimal offline server will be able to visit all cities by time $2n+a$. Therefore, noting that $Z^{r=0}(n,m)=2n/m$, we have that

$$\begin{aligned}\frac{Z^A(n,m)}{Z^*(n,1)}&\ge\frac{(1+1/\gamma))(2n/m-1)}{2n+a}\\&=(1+1/\gamma)(1/m)\frac{Z^{r=0}(n,m)}{Z^{r=0}(n,m)+a/m}-\frac{1+1/\gamma}{2n+a}\\&=(1+1/\gamma)(1/m)\frac{1}{1+\alpha/m}-\frac{1+1/\gamma}{2n+a};\end{aligned}$$

taking n arbitrarily large proves the theorem. $\square$

Proof of Theorem 10. The proof is very similar to that of Theorem 1; we detail only the differences for the case analysis.

Case (1): The time needed by PAH-G is at most $r_n + \rho Z^{r=0}(n,Q)/\gamma \leq Z^*(n,q) + \rho Z^{r=0}(n,Q)/\gamma$. By Lemma 3, this upper bound is asymptotically equal to $Z^*(n,q) + \frac{\rho q}{\gamma Q} Z^{r=0}(n,q) \leq (1 + \frac{\rho q}{\gamma Q}) Z^*(n,q)$, almost surely.

Case (2a): Applying Lemma 3, we have that the online cost is almost surely at most

$$\begin{aligned} r_n + \frac{d(o,l_n^*)}{\gamma} + \rho\frac{Z^{r=0}(n,Q)}{\gamma} &\leq Z^*(n,q) + \left(\frac{2\rho-\gamma+1}{2\gamma}\right) Z^{r=0}(n,Q) \\ &\to Z^*(n,q) + \left(\frac{2\rho-\gamma+1}{2\gamma}\right)\left(\frac{q}{Q}\right) Z^{r=0}(n,q) \\ &\leq \left(1 + \left(\frac{2\rho-\gamma+1}{2\gamma}\right)\left(\frac{q}{Q}\right)\right) Z^*(n,q) \end{aligned}$$

Case (2b): Since $Z^*(n,Q) \leq Z^*(n,q)$, the online cost is almost surely at most

$$\begin{aligned} Z^*(n,Q) + \frac{\rho}{\gamma} Z^{r=0}(n,Q) + \left(\frac{\rho-1}{\gamma}\right) Z^*(n,Q) &\leq Z^*(n,q) + \frac{\rho}{\gamma} Z^{r=0}(n,Q) + \left(\frac{\rho-1}{\gamma}\right) Z^*(n,q) \\ &\to Z^*(n,q) + \frac{\rho q}{\gamma Q} Z^{r=0}(n,q) + \left(\frac{\rho-1}{\gamma}\right) Z^*(n,q) \\ &\leq \left(1 + \frac{\rho q}{\gamma Q} + \frac{\rho-1}{\gamma}\right) Z^*(n,q). \end{aligned}$$

Since $\rho, \gamma \geq 1$ and $Q \geq q \geq 0$, $\max\left\{1 + \frac{\rho q}{\gamma Q}, 1 + \frac{\rho q}{\gamma Q} + \frac{\rho-1}{\gamma}, 1 + \left(\frac{2\rho-\gamma+1}{2\gamma}\right)\left(\frac{q}{Q}\right)\right\} = 1 + \frac{\rho q}{\gamma Q} + \frac{\rho-1}{\gamma}$ and the theorem is proved. □