Market Design for Dynamic Pricing and Pooling in Capacitated Networks

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Abstract

We study a market mechanism that sets edge prices to incentivize strategic agents to organize trips that efficiently share limited network capacity. This market allows agents to form groups to share trips, make decisions on departure times and route choices, and make payments to cover edge prices and other costs. We develop a new approach to analyze the existence and computation of market equilibrium, building on theories of combinatorial auctions and dynamic network flows. Our approach tackles the challenges in market equilibrium characterization arising from: (a) integer and network constraints on the dynamic flow of trips in sharing limited edge capacity; (b) heterogeneous and private preferences of strategic agents. We provide sufficient conditions on the network topology and agents' preferences that ensure the existence and polynomial-time computation of market equilibrium. We identify a particular market equilibrium that achieves maximum utilities for all agents, and is equivalent to the outcome of the classical Vickery Clark Grove mechanism. Finally, we extend our results to general networks with multiple populations and apply them to compute dynamic tolls for efficient carpooling in San Francisco Bay Area.

1 Introduction

Transportation and logistics networks often face losses from congestion caused by inefficient use of limited capacity. One effective approach to improve resource utilization and reduce costs is to incentivize users to share the network capacity by resource pooling. This can be done through a market-based approach that sets appropriate prices for capacity usage at specific times and locations. For example, setting edge toll prices for using road networks can incentivize agents to organize carpooled trips, reduce the number of vehicles on the road, and minimize congestion. Similarly, flexible and on-demand shipping applications can optimize deliveries and routes, reducing the number of cargoes and the fleet size required. By leveraging the complementarity between *resource pooling* and *capacity-based pricing*, we can achieve substantial improvement in cost, time, and reducing environmental impacts for transportation and logistics networks.

Our main contributions. The goal of this paper is to build a dynamic market mechanism that incentivizes strategic agents to share network capacities by forming groups and share trips. We consider the setting with a discrete and finite time horizon. The transportation network has finite edge capacity, restricting the maximum number of trips that can enter each edge at each time step. A trip is determined by the group of agents who share a vehicle, the departure time, and the taken route. Agents have *private and heterogeneous* trip preferences that incorporate their preferred latest arrival time, cost of late arrival, sensitivity to travel time of the taken route, and the disutility of trip sharing that depends on the group size.

Each trip, when entering an edge at a particular time, is charged with a price for occupying one unit capacity at that time step. The price of each edge is dynamic, and serves as the "invisible hand" of the market that governs agents' group formation and the spatiotemporal distribution of demand of capacity – when the price of an edge increases, agents are more incentivized to form groups to share trips and split the price or to change a route and a departure time. Agents in each group make payments to cover the edge prices and other trip costs.

A market equilibrium is defined as the trip organization, dynamic edge prices, and agent payments that satisfy four important conditions: *(i)* individual rationality – all agents have non-negative equilibrium utility; *(ii)* stability – no group of agents have incentive to deviate from the equilibrium trips; *(iii)* budget balance – agents' payments can cover the trip cost and edge capacity prices; *(iv)* market clearing – prices are only charged on edges and time steps such that the edge capacity is saturated. When equilibrium exists, the equilibrium trip organization is *feasible* – the number of vehicles that enter each edge at each time step does not exceed the edge capacity, and *socially optimal* – maximizes the total value of all organized trips (Ostrovsky and Schwarz [2019]).

Our main contributions include: (i) characterizing sufficient conditions (tractable scenarios) under which market equilibrium is guaranteed to exist, and can be computed in polynomial time (Sections 3 – 4); (ii) extending the results and algorithms beyond the tractable scenarios to study organization of trip sharing markets in general networks with multiple origin-destination pairs; (iii) using carpooling market as an example, we demonstrate the applicability of our approach by computing the dynamic edge pricing (tolling) and carpool trips for the San Francisco Bay area highway network (Section 5).

Our first step of analysis is to construct an integer program that solves the socially optimal integer trip organization problem, and demonstrate that market equilibrium existence is equivalent to zero integrality gap of the associated linear relaxation problem (Proposition 1). This result converts the equilibrium existence problem to the existence of integer optimal solution in the linearly relaxed program. However, such linear relaxation is not directly useful even for computing the quasi- market equilibrium (i.e., equilibrium that drops integer constraints or assumes a large market limit) due to its trip organization variables being exponential in the number of agents and edges. Therefore, it is crucial to identify conditions under which equilibrium computation is tractable, and provide efficient algorithms.

We develop a new approach for analyzing market equilibrium by leveraging ideas from the *dynamic network flow theory* (Skutella [2009], Hoppe and Tardos [2000], Ruzika et al. [2011]) and the *theory of combinatorial auction* (Kelso Jr and Crawford [1982], Gul and Stacchetti [1999], De Vries and Vohra [2003], Leme [2017], Feldman et al. [2013]). This approach is natural for our setting since the trip organization involves determining the dynamic flow of vehicles in the network, and incentivizing agents with heterogeneous preferences to form group coalitions. The interesting aspect of this approach comes from the interaction between the dynamic flow of vehicles and the trip sharing incentives – agents' incentives of forming groups are impacted by the departure time and taken route of the trip due to their heterogeneous sensitivities of travel time and preferred latest arrival times.

We show that the sufficient conditions for market equilibrium existence include both the condition on network topology – being *series-parallel*, and the condition on agents' preferences – having *homogeneous disutilities of sharing trips* (Theorem 1). We demonstrate (some level of) "tightness" of the two sufficient conditions by providing two simple counter examples (Examples 1 and 2) that each violates one of the two conditions and market equilibrium fails to exist. These conditions are not necessary conditions since any linear programs may coincidentally have integer optimal solutions. We show in Sec. 4 that these conditions also play a crucial role in developing polynomial-time algorithms for computing market equilibrium.

In our proof of Theorem 1, we show that in series-parallel networks, there exists an optimal solution of the relaxed linear program such that the induced route flow is an *earliest arrival flow* of the network, which takes integer value for each route and each departure time. However, this property is no longer guaranteed for non series-parallel networks, which leads to optimal trip flow being fractional. To see why the network topology condition plays an important role in equilibrium existence, we notice that any trip that takes a certain route utilizes a unit capacity of all edges on that route, and the edge price on one edge affects the capacity utilization of all edges that share a route with it. Thus, both the trip organization and edge prices are crucially influenced by the network structure.

Additionally, the homogeneous disutility of trip sharing condition stems from the coalition formation. We show that forming groups with integer capacities is *mathematically equivalent* to a Walrasian equilibrium in an auxiliary economy, with agents as "indivisible goods" and route capacity units as "agents" with an *augmented trip value function*. The condition of homogeneous disutility of trip sharing ensures that the augmented trip value function satisfies the gross substitutes condition, even though the original value function does not. Consequently, Walrasian equilibrium exists in the auxiliary economy (Gul and Stacchetti [1999]), which can be turned into a market equilibrium in our setting.

Furthermore, we identify a particular market equilibrium such that the trip organization and payment are identical to that of a Vickery Clark Grove mechanism. Interestingly, this equilibrium also has the advantage of achieving the highest agent utilities among all market equilibria, and only collecting the minimum total edge prices (Theorem 2). We also develop a two-step polynomial-time algorithm to compute the market equilibrium. The design of our algorithm builds on the proofs of Theorems 1 – 2: We first compute the equilibrium route flow capacity as the earliest arrival flow in the series-parallel network using a greedy algorithm (Algorithm 1). Then, we compute equilibrium groups given the route flow capacity as the Walrasian equilibrium in the equivalent economy with augmented trip functions (Algorithm 3). In this algorithm, we develop an efficient way to iteratively compute the augmented trip value functions, and build on the well-known Kelso-Crawford algorithm (Kelso Jr and Crawford [1982]) that exploits the gross substitutes condition provided by the homogeneous disutlities of trip sharing.

In Section 5, we extend our results to general networks with multiple origin-destination pairs, and agents are separated into multiple populations with different disutility of sharing trips. Building on Theorem 1, we show that market equilibrium still exists if capacity prices are set on routes rather than edges, and each agent population is served in a separate sub-market. Practically, different sub-markets correspond to customers with different trip flexibility and sharing preferences, and can be served with different departure frequencies and by fleet of different sizes. Our two-step algorithms can be used to compute equilibrium in each sub-market. The social welfare in equilibrium depends on how the edge capacities are allocated across different sub-markets in each time step.

We show that the optimal allocation of network capacity across different sub-markets is an NP hard problem even if the network is series-parallel (Proposition 3), and provide a lower bound of integrality gap in Proposition 4. We formulate the socially optimal capacity allocation problem as an integer program, and provide a branch and price algorithm (Algorithm 4). In each iteration of the algorithm, the polynomial-time computation of the sub-market equilibrium again builds on Theorem 1 and the fact that agents in each sub-market have identical disutilities of sharing trips.

Related literature: The paper Ostrovsky and Schwarz [2019] has proposed a competitive market framework for sharing network capacity in the context of autonomous carpooling. In this paper, authors defined the concept of market equilibrium in static settings, and demonstrate that a market equilibrium (when exists) maximizes the social welfare. Our work builds on Ostrovsky and Schwarz [2019] and extends to the dynamic setting, where agents' preferences of trip organization depend on their latest arrival time, delay cost, sensitivity to travel time, and disutilities of trip sharing. We further study the equilibrium existence, computation, and demonstrate its applicability in real-world settings.

Market design approach has been widely adopted in many applications that involve network constraints with limited capacity. Kelly [1997], Johari and Tsitsiklis [2004], Yang and Hajek [2007], Jain and Walrand [2010] studied pricing and bandwidth sharing in communication networks with both price taking agents and price anticipating agents. In this application, agents make payments to use a fraction of link capacity, and their utilities depend on the allocated capacity and the price. Another important application of network market design is spectrum auctions (Cramton [1998], Berry et al. [2010], Newman et al. [2017]). This line of work concentrates on devising market mechanisms and auction procedures to tackle efficiency and market-clearing challenges that arise from network constraints and externalities among bidders. We consider a competitive market framework in our problem. Agents are not allocated with or bidding for a certain fraction of capacity, but rather cooperatively form coalitions to share a trip based on their heterogeneous preferences.

Our work is also related to the growing literature on the operations of mobility-on-demand services and transportation marketplaces. In particular, the studies of the mobility-on-demand services have focused on the spatial and temporal pricing (e.g. Banerjee et al. [2015], Castillo et al. [2017], Bimpikis et al. [2019], Yan et al. [2020], Besbes et al. [2021], Garg and Nazerzadeh [2022], Ma et al. [2022], Freund and van Ryzin [2021]), and dynamic matching (e.g. Ashlagi et al. [2019], Özkan and Ward [2020], Castro et al. [2021], Gurvich and Ward [2015], Afeche et al. [2018], Feng et al. [2021], Hu and Zhou [2022]). Moreover, recent literature has focused on the operations of carpooling services that include the study of ride matching and route planning Alonso-Mora et al. [2017], Santi et al. [2014], Stiglic et al. [2015], Ashlagi et al. [2019], Pavone et al. [2022], Lobel and Martin [2020], Taylor [2023], Zhang et al. [2023], and pricing Hu et al. [2020], Jacob and Roet-Green [2021], Zhang and Nie [2021]. A key challenge in ride-hailing market problems comes from the two-sided nature of matching between agents (demand) and drivers (supply), and prices are set on services for balancing the incentives of both sides. Our paper departs from this line of literature by focusing on setting prices on the limited physical network capacity rather than the services directly. How the capacity price is split as payments of each agent (which can be viewed as the price of service (trip) that they receive) depends on agents' heterogeneous preferences of various factors such as value of time, preferred arrival time, and their sensitivity to trip sharing. As a result, the key challenge of our analysis arises from analyzing how the prices impact the group formation among agents with heterogeneous preferences under physical constraints imposed by car size, edge capacity, and network structure, not the matching between riders and drivers.

2 A Market Model

2.1 Network, Agents, and Trips

A finite set of agents $m = 1, \dots, M$ organize trips in a network at discrete time steps $t \in \{1, 2, \dots, T\}$. We model the network as a directed graph. We present our main results for networks with single origin-destination pair, and provide extensions to general networks in Section 5. The set of edges in the network is E. The capacity of each edge $e \in E$ is a positive integer $q_e \in \mathbb{N}_+$ that represents the maximum number of trips that can enter the edge at each time step t. The set of routes is R, where each route $r \in R$ is a sequence of edges that form a directed path from the origin to the destination. We denote the travel time of each edge e as $d_e > 0$, and the travel time of each route r as $d_r = \sum_{e \in r} d_e$.

A *trip* is defined as a tuple (z, b, r), where $z \in \{1, 2, \dots, T\}$ is the departure time at the origin, $b \in B \triangleq \left\{2^M \mid |b| \leq A\right\}$ is a group of agents who share the trip with maximum group size of A, and $r \in R$ is the route that the trip takes. A trip is *feasible* if the arrival time at the destination is before time T. The set of all feasible trips is given by:

$$Trip \triangleq \left\{ (z,b,r) \;\middle|\; \begin{array}{l} z = 1,2,\dots T,\ r \in R, \\ z + d_r \leq T,\ b \in B \end{array} \right\}. \tag{1}$$

For each agent $m \in M$, the value of a trip (z, b, r) such that $b \ni m$ is given by:

$$v_{m,r}^{z}(b) = \alpha_m - \beta_m d_r - \pi_m(|b|) - \gamma_m(|b|) d_r - \ell_m((z + d_r - \theta_m)_+). \tag{2}$$

where

- α_m is agent m's value of arriving at the destination.
- β_m is agent m's value of time. When taking route r, the disutility of spending time d_r is $\beta_m d_r$.
- $\pi_m(|b|)+\gamma_m(|b|)d_r$ is agent m's disutility of sharing a trip with size $|b|$, and the disutility is linear in the time cost d_r.
- $\ell_m((z + d_r - \theta_m)_+)$ is agent m's cost of delay, where θ_m is agent m's preferred latest arriving time, and $(z + d_r - \theta_m)_+ = \max\{z + d_r - \theta_m, 0\}$ is the time of agent m being late. The function $\ell_m : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ can be any non-decreasing function, and $\ell_m(0) = 0$ for all $m \in M$.

We note that the disutility of sharing a trip only depends on the group size $|b|$ rather than the identify of agents in the group. We consider that $\pi_m(|b|), \gamma_m(|b|) \geq 0$ for all $|b| = 1, \dots, A$, and the disutility of sharing a trip is zero for any solo trip, i.e. $\pi_m(1), \gamma_m(1) = 0$ for all $m \in M$. That is, agents prefer to take solo trips rather than sharing with others, and the disutility increases with the travel time. Additionally, we assume that the marginal disutilities $\pi_m(|b| + 1) - \pi_m(|b|)$ and $\gamma_m(|b| + 1) - \gamma_m(|b|)$ are non-decreasing in the group size $|b|$ for all $|b| = 1, \dots, A-1$. Therefore, disutilities of all agents are non-decreasing in the group size, and the extra disutility of adding one agent to any trip (z, b, r) is non-decreasing in the group size $|b|$.

The cost of a trip with group b and route r equals to $c_r(b) = (\sigma + \delta d_r)\,|b|$. The cost is non-negative, and increases with the trip time, i.e. $\sigma, \delta \geq 0$. The total value of each trip (z, b, r) is the summation of the trip values for all agents in b net the trip cost:

$$V_r^z(b) = \sum_{m \in b} v_{m,r}^{z}(b) - c_r(b). \tag{3}$$

2.2 Competitive market equilibrium

We represent the market outcome by the tuple (x, p, τ), where x is the trip organization vector, $p = (p_m)_{m\in M}$ is the payment vector, where p_m is the payment charged from agent m, and $\tau = (\tau_e^t)_{e\in E, t=1,\dots,T} \in \mathbb{R}_{\geq 0}^{|E|\times T}$ is the edge price vector, where τ_e^t is the edge price for a trip that enters edge e at time t. Given the edge price vector τ, the price for a trip (z, b, r) equals to $\sum_{e\in r} \tau_e^{z+d_{r,e}}$, where $d_{r,e}$ is the time cost from the origin to the beginning of edge e along the route r, and $\tau_e^{z+d_{r,e}}$ is the price of edge e when the vehicle enters the edge at time $z + d_{r,e}$. The trip organization vector is $x = (x_r^z(b))_{(z,r,b)\in Trip} \in \{0, 1\}^{|Trip|}$, where $x_r^z(b) = 1$ if trip (z, b, r) is organized and $x_r^z(b) = 0$ if otherwise. A feasible trip vector x must satisfy the following constraints:

$$\sum_{(z,r,b)\in\{Trip|b\ni m\}} x_r^z(b) \leq 1, \quad \forall m \in M, \tag{4a}$$

$$\sum_{(z,r,b)\in Trip} x_r^{t-d_{r,e}}(b) \leq q_e, \quad \forall e \in E, \quad \forall t = 1, \dots, T, \tag{4b}$$

$$x_r^z(b) \in \{0, 1\}, \quad \forall (z, r, b) \in Trip. \tag{4c}$$

where (4a) ensures that no agent takes more than 1 trip, and (4b) ensures that the total number of trips that enter any edge $e \in E$ at any time t does not exceed the edge capacity q_e.

Given any (x, p, τ), the utility of each agent $m \in M$ equals to the value of the trip that m takes minus the payment:

$$u_m = \sum_{(z,r,b)\in\{Trip|b\ni m\}} v_{m,r}^z(b) x_r^z(b) - p_m, \quad \forall m \in M. \tag{5}$$

We define the market equilibrium as an outcome (x^*, p^*, τ^*) that satisfies four properties – *individual rationality*, *stability*, *budget balance*, and *market clearing*.

Definition 1. A market outcome (x^*, p^*, τ^*) is an equilibrium if it satisfies

1. *Individually rationality:* All agents' utilities u^* as in (5) are non-negative, i.e.

$$u_m^* \geq 0, \quad \forall m \in M. \tag{6}$$

2. *Stability:* No agent group in B can gain higher total utility by organizing a different

trip:

$$\sum_{m \in b} u_m^* \geq V_r^z(b) - \sum_{e \in r} \tau_e^{z+d_{r,e}*}, \quad \forall (z, r, b) \in Trip. \tag{7}$$

3. *Budget balance:* The total payments of each organized trip equals to the sum of the edge prices and the cost of the trip. An agent's payment is zero if they are not part of any organized trip:

$$x_r^{z*}(b) = 1, \quad \Rightarrow \quad \sum_{m \in b} p_m^* = \sum_{e \in r} \tau_e^{z+d_{r,e}*} + c_r(b), \quad \forall (z, r, b) \in Trip, \tag{8a}$$

$$x_r^{z*}(b) = 0, \quad \forall (z, r, b) \in \{Trip | b \ni m\}, \quad \Rightarrow \quad p_m^* = 0, \quad \forall m \in M. \tag{8b}$$

4. *Market clearing:* For any edge $e \in E$ and any time $t = 1, \ldots, T$, the edge price τ_e^{t*} is zero when the number of trips entering edge e at time t is below the edge capacity:

$$\sum_{(z,r,b) \in \{Trip | r \ni e\}} x_r^{t-d_{r,e}*}(b) < q_e, \quad \Rightarrow \quad \tau_e^{t*} = 0, \quad \forall e \in E, \quad \forall t = 1, \ldots, T. \tag{9}$$

In Definition 1, individual rationality condition (6) prevents agents from opting out of the market. The stability condition in (7) ensures that the total utility $\sum_{m \in b} u_m^*$ for any group b induced by the market equilibrium (x^*, p^*, τ^*) is higher or equal to the maximum total utility that can be obtained by b taking any feasible trip (z, b, r), which is the trip value $V_r^z(b)$ minus the edge prices $\sum_{e \in r} \tau_e^{z+d_{r,e}*}$. Thus, agents have no incentive to form another trip not in the equilibrium. The individual rationality condition and the stability condition together guarantee that agents will follow the equilibrium trip organization. The budget balance condition guarantees that the payments cover edge prices and trip costs, while market clearing ensures that non-zero equilibrium edge prices only on fully utilized edges and time stages.

2.3 Primal and Dual Formulations

The following integer program solves the socially optimal trip vector that maximizes the total social welfare $S(x)$ of all organized trips:

$$\max_x \quad S(x) = \sum_{(z,r,b) \in Trip} V_r^z(b) x_r^z(b), \quad s.t. \quad x \text{ satisfies (4a) – (4c).} \tag{IP}$$

We introduce the linear relaxation of (IP) as the primal linear program:

$$\max_{x} \quad S(x) = \sum_{(z,r,b)\in Trip} V_r^z(b) x_r^z(b),$$

$$s.t. \quad \sum_{(z,r,b)\in \{Trip|b\ni m\}} x_r^z(b) \le 1, \quad \forall m \in M, \tag{LP.a}$$

$$\sum_{(z,r,b)\in \{Trip\}} x_r^{t-d_{r,e}}(b) \le q_e, \quad \forall e \in E, \quad \forall t = 1, \dots, T, \tag{LP.b}$$

$$x_r^z(b) \ge 0, \quad \forall (z,r,b) \in Trip. \tag{LP.c}$$

Note that the constraint $x_r^z(b) \le 1$ is implicitly included in (LP.a), and thus is omitted.

By introducing dual variables $u = (u_m)_{m\in M}$ for constraints (LP.a) and $\tau = (\tau_e^t)_{e\in E, t=1,\dots,T}$ for constraints (LP.b), the dual of (LP) can be written as follows:

$$\min_{u,\tau} \quad U(u,\tau) = \sum_{m\in M} u_m + \sum_{t=1}^{T} \sum_{e\in E} q_e \tau_e^t,$$

$$s.t. \quad \sum_{m\in b} u_m + \sum_{e\in r} \tau_e^{z+d_{r,e}} \ge V_r^z(b), \quad \forall (z,r,b) \in Trip, \tag{D.a}$$

$$u_m \ge 0, \quad \tau_e^t \ge 0, \quad \forall m \in M, \quad \forall e \in E, \quad \forall t = 1, \dots, T. \tag{D.b}$$

The dual variables $u = (u_m)_{m\in M}$ and $\tau = (\tau_e^z)_{e\in E, z=1,\dots,T}$ can be viewed as agents' utilities and the edge prices, respectively. In (D), the objective $U(u,\tau)$ equals the sum of all agents' utilities and the total collected edge prices, and (D.a) is the same as the stability condition in (7).

Proposition 1. *A market equilibrium (x^*, p^*, τ^*) exists if and only if* (LP) *has an optimal integer solution. Any optimal integer solution x^* of* (LP) *is an equilibrium trip vector, and any optimal solution (u^*, τ^*) of* (D) *is an equilibrium utility vector and an equilibrium edge price vector. The equilibrium payment vector p^* is given by:*

$$p_m^* = \sum_{r\in R} \sum_{z=1}^{T} \sum_{b\ni m} x_r^*(b) v_{m,r}^z(b) - u_m^*, \quad \forall m \in M. \tag{12}$$

The primal-dual formulation in Proposition 1 enables us to convert the market equilibrium existence problem to the existence of integer optimal solution in (LP). In the proof of Proposition 1, we show that the four properties of market equilibrium – individual rationality, stability, budget balance, and market clearing – are equivalent to the feasibility constraints and the complementary slackness conditions in (LP) and (D). Following from strong duality, a market equilibrium exists if and only if the optimality gap between the linear relaxation (LP) and the integer problem (IP) is zero. That is, the existence of market equilibrium is equivalent to the existence of integer optimal solutions in (LP).

Even if we ignore the integer constraints, the linear programs (LP) and (D) cannot be directly used to compute market equilibrium because the primal (resp. dual) program has exponential number of variables (resp. constraints). Moreover, the integrality gap of (LP) can be significant as we will show by examples in Section 3 and by the bound of the integrality gap in Section 5. When (LP) does not have an integral solution, Proposition 1 indicates that even if (IP) is solved, there does not exists an edge price vector and a payment vector to implement the optimal trip organization so that agents are willing to take those trips.

In Sections 3 – 4, we provide conditions that guarantee the existence and tractability of integer solution in (LP), and provide a polynomial time algorithm to compute the market equilibrium. We extend the results of the tractable case to general networks with multiple origin-destination pairs in Section 5.

3 Existence and properties of market equilibrium

In Sec. 3.1, we characterize sufficient conditions that guarantee market equilibrium existence. In Sec. 3.2, we identify a market equilibrium that is equivalent to the outcome of the classical Vickery Clark Groves mechanism, and this equilibrium achieves the maximum utility of each player among all market equilibria.

3.1 Sufficient conditions for equilibrium existence

In this section, we characterize the sufficient conditions on network topology and trip values under which there exists a market equilibrium. Before we present the results, we first introduce the definition of series-parallel network:

Definition 2 (Series-Parallel (SP) Network Milchtaich [2006])**.** A network is series-parallel if it is constructed by connecting two series-parallel networks either in series or in parallel for finitely many iterations. Equivalently, a network is series-parallel if and only if a wheatstone structure as Figure 1 is not embedded.

Theorem 1. *Market equilibrium (x^*, p^*, τ^*) exists if the network is series-parallel, and agents have homogeneous disutilities of trip sharing, i.e.*

$$\pi_m(d) = \pi(d), \quad \gamma_m(d) = \gamma(d), \quad \forall d = 1, \ldots, A, \quad \forall m \in M. \tag{13}$$

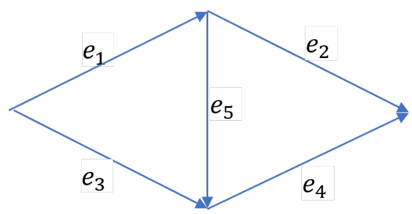

Figure 1: Wheatstone network

Theorem 1 shows that the sufficient condition for equilibrium existence includes both the condition on network topology – being series-parallel– and the condition on trip sharing disutiliy parameters – being identical. We note that the homogeneous trip sharing distutility condition still allows agents to have heterogeneous trip values – agents can have different values of arriving at the destination α_m, value of times β_m, preferred latest arrival times θ_m, and late-arrival costs ℓ_m.

We next provide two counterexamples, where market equilibrium does not exist when either one of the two sufficient conditions in Theorem 1 does not hold.

Example 1. Consider the wheatstone network as in Figure 1. The capacity of each edge in the set $\{e_1, e_2, e_3, e_4\}$ is 1, and the capacity of edge e_5 is 4. The travel time of each edge is given by $d_1 = 1$, $d_2 = 2$, $d_3 = 2$, $d_4 = 1$, and $d_5 = 0.2$.

The maximum capacity of vehicle is $A = 2$. Three agents $m = 1, 2, 3$ travel on this network. agents have identical preference parameters: value of trip $\alpha_m = 6$, value of time $\beta_m = 1$, zero trip sharing disutility, i.e. $\pi_m(|b|) = 0$ and $\gamma_m(|b|) = 0$ for any $|b| = 1, 2$ and any $m \in M$. The latest arrival time for all agents is 4, and the delay cost of arriving after $t = 4$ is infinity. For simplicity, we set the trip cost parameters as zero in this example, i.e. $\sigma = 0$, $\delta = 0$.

We note that any trip that departs at time $z \geq 2$ has zero value since the arrival time is later than 4. Thus, we only need to consider trips with $z = 1$. We define route e_1-e_2 as r_1, e_1-e_5-e_4 as r_2, and e_3-e_4 as r_3. Trip values are: $V_1^1(m) = V_3^1(m) = 3$, and $V_2^1(m) = 3.8$ for all $m \in M$; $V_1^1(m, m') = V_3^1(m, m') = 6$, and $V_2^1(m, m') = 7.6$ for all $m, m' \in M$. The unique optimal solution of the linear program (LP) is $x_1^{1*}(1, 2) = x_2^{1*}(2, 3) = x_3^{1*}(1, 3) = 0.5$, and $S(x^*) = 9.8$. That is, (LP) does not have an integer optimal solution, and market equilibrium does not exist (Proposition 1).

Example 2. Consider a network with two parallel edges e_1, e_2. Both edges have a capacity of 1 and a travel time of $d_1 = d_2 = 1$. The maximum capacity of a vehicle is $A = 6$. Twelve agents travel on this network; the latest arrival time for all agents is 3, and the delay cost of arriving after $t = 3$ is infinity. Furthermore $\gamma_m(|b|) = 0$ for any $|b| = 1, 2, 3, 4, 5, 6$, and any $m \in M$. Agents $1, 2, \ldots, 6$ have the following preference parameters: the value of arriving at the destination $\alpha_m = 50$, value of time $\beta_m = 1/6$, and disutility of trip sharing is $\pi_m(|b|) = 0.25(|b| - 1)$ for $|b| \leq 5$, and $0.5(n-1)$ for $|b| = 6$. Agents $7, 8, \ldots, 12$ have a value of arriving at the destination $\alpha_m = 100$, $\beta_m = 0.5$, and their disutility of trip sharing is $\pi_m(|b|) = 2(|b| - 1)$ if $|b| \leq 4$ and infinity otherwise.

The optimal solution to the LP-relaxation is $x^1_{e_1}(\{1, 2, 3, 4, 5, 6\}) = 0.5$, $x^1_{e_1}(\{9, 10, 11, 12\}) = 0.5$, $x^1_{e_2}(\{7, 8, 10, 12\}) = 0.5$, $x^1_{e_2}(\{7, 8, 9, 11\}) = 0.5$. This solution has a value of 662.5. The optimal integer solution schedules the trip $\{1, 2, 3, 4, 5, 6\}$ at time 1 on e_1, and $\{9, 10, 11, 12\}$ at time 1 on e_2; this solution has value of $621 < 662.5$. This indicates that the LP relaxation does not have an integer optimal solution, and thus market equilibrium does not exist.

Theorem 1 indicates that the network topology plays a crucial role in the stability and efficiency of resource sharing in the market. Although most road networks in practice are not series-parallel, services and operations such as carpool lanes and shipment routes are often set on a subset of routes that have a much simpler network topology. Moreover, the condition of identical disutility of trip sharing indicates that agents with different trip sharing disutilities should be separated into different markets. In Section 4, we further show that *the homogeneous disutility condition is a necessary condition for polynomial time computation of market equilibrium.*

The two sufficient conditions in Theorem 1 provide valuable insights for designing the trip market in general networks with multiple origin-destination pairs and agent populations with heterogeneous trip sharing disutilities. As we will show in more details in Section 5, the market design in the general setting involves first creating separate market for each population that has the same origin-destination pair and the same trip sharing disutility, then allocating edge capacities to each sub-market, where equilibrium is guaranteed to exist.

For the rest of this section, we present the proof sketch of Theorem 1. The complete proof is included in Section C.

Proof sketch. The proof of Theorem 1 is built on ideas from the theory of *earlist arrival network flow problem* (Skutella [2009], Hoppe and Tardos [2000], Ruzika et al. [2011]) and the *combinatorial auction theory* (Kelso Jr and Crawford [1982], Gul and Stacchetti [1999]). Recall from Proposition 1 that showing the existence of market equilibrium is equivalent to proving that (LP) has an integer optimal solution. In step 1 of the theorem proof, we show that when network is series-parallel, there exists an optimal solution of (LP) such that the total flow of trips that take each route in each time step is integer, and such flow vector can be computed as the earliest arrival flow of the network. In step 2, we show that with homogeneous trip sharing disutilites, the optimal trip sharing group formation that satisfies the constructed flow constraints in step 1 is also integer.

Step 1. We construct a dynamic flow capacity vector $k^* = (k_r^{z*})_{r \in R, z=1,\ldots,T}$ that sets capacity w_r^* of taking route r at each feasible time step z:

$$k_r^{z*} = w_r^*, \quad \forall r \in R, \quad \forall z = 1, 2, \ldots, T - d_r. \tag{14}$$

where $w^* = (w_r^*)_{r \in R}$ is computed by the greedy Algorithm 1 that allocates edge capacity $(q_e)_{e \in E}$ to routes in increasing order of travel time. In this algorithm, we begin with computing a shortest route r_{min} with travel time d_{min}, and sets its capacity to be the maximum possible capacity $w^*_{r_{min}} = \min_{e \in r_{min}} q_e$. Then, we reduce the residual capacity of each edge on r_{min} by $w^*_{r_{min}}$, and repeat this process until there exists no route with positive residual capacity in the network. The dynamic flow capacity vector k^* is the temporally repeated flow that allocates w_r^* capacity to route r for every feasible departure time $z = 1, 2, \ldots, T - d_r$. We denote $R^* = \{R | w_r^* > 0\}$ as the set of routes with positive flow capacity.

ALGORITHM 1: Greedy algorithm for computing static route capacity w^*

Initialize: Set $\tilde{E} \leftarrow E$; $\tilde{q}_e \leftarrow q_e$, $\forall e \in \tilde{E}$; $w_r^* \leftarrow 0$, $\forall r \in R$;
$(d_{min}, r_{min}) \leftarrow ShortestRoute(\tilde{E})$;
while $d_{min} < \infty$ **do**
  $w^*_{r_{min}} \leftarrow \min_{e \in r_{min}} \tilde{q}_e$;
  for $e \in r_{min}$ **do**
    $\tilde{q}_e \leftarrow \tilde{q}_e - w^*_{r_{min}}$;
    if $\tilde{q}_e = 0$ **then**
      $\tilde{E} \leftarrow \tilde{E} \setminus \{e\}$;
    end
  end
  $(d_{min}, r_{min}) \leftarrow ShortestRoute(\tilde{E})$;
end
Return w^*

We consider another socially optimal trip organization problem (LPk^*), where trips satisfy the capacity constraints according to k^*. Problem (LPk^*) is more restrictive than the original problem (LP), as trip vectors satisfying capacity constraints in (LPk^*.b) must also meet the original network capacity constraint (LP.b), but not necessarily vice versa. Lemma 1 demonstrates that for series-parallel networks, an optimal solution of (LPk^*) also optimizes the original problem (LP).

Lemma 1. *If the network is series-parallel, then any optimal solution of* (LPk^*) *is an* *optimal solution of* (LP)*:*

$$
\begin{aligned}
\max_{x} \quad & S(x) = \sum_{(z,r,b)\in Trip} V_r^z(b) x_r^z(b) \\
s.t. \quad & \sum_{(z,r,b)\in\{Trip|b\ni m\}} x_r^z(b) \leq 1, \quad \forall m \in M, && \text{(LP}k^*\text{.a)} \\
& \sum_{b\in B} x_r^z(b) \leq k_r^{z*}, \quad \forall r \in R, \quad \forall z = 1,\ldots,T-d_r, && \text{(LP}k^*\text{.b)} \\
& x_r^z(b) \geq 0, \quad \forall b \in B, \quad \forall r \in R, \quad \forall z = 1,2,\ldots,T. && \text{(LP}k^*\text{.c)}
\end{aligned}
$$

We prove Lemma 1 by construction. We show that on a series-parallel network, for any feasible solution x of (LP) on a series-parallel network, we can construct another trip vector $\hat{x}$ satisfying $S(\hat{x}) \geq S(x)$ and feasibility in (LPk^*). Optimal values of (LPk^*) and (LP) are equal, making any optimal solution of (LPk^*) optimal in (LP).

The key step of the proof is to construct such $\hat{x}$ by redistributing flow of agent groups in x, ensuring no group has later arrival time in $\hat{x}$ compared to x and that agent groups with higher time sensitivity are prioritized for shorter routes. The series-parallel network condition is used to show that the temporally repeated flow k^* as in (14) is the earliest arrival flow in series-parallel networks (Lemma 8 in Appendix C). Thus, $\hat{x}$ has the same total flow of each b as in x, and the flow arriving before each time step t given $\hat{x}$ is no less than that in x. We also prove, using mathematical induction, that $\hat{x}$ has higher social welfare compared to x (i.e. $S(\hat{x}) \geq S(x)$) when the network is series-parallel: If the inequality holds on any two series-parallel networks, then it also holds on the network constructed by connecting the two sub-networks in series or in parallel.

Part 2. In this part, we show that when agents have homogeneous trip sharing disutilities, (LPk^*) has an integer optimal solution. Following from Lemma 1 in part 1, we know that this solution is also an optimal integer solution of (LP), and thus conclude Theorem 1. In this step, we need to introduce the definitions of monotonicity and gross substitutes.

Definition 3 (Monotonicity)**.** A function $f : B \to \mathbb{R}$ is monotone if $f(b \cup b') \geq f(b)$, $\forall b, b' \in B$.

The value of a monotonic function f increases as the set b increases.

Definition 4 (Gross Substitutes Reijnierse et al. [2002])**.** A function $f : B \to \mathbb{R}$ satisfies the gross substitutes condition if

(i) $\forall b, b' \subseteq B$ such that $b \subseteq b'$ and any $i \in M \setminus b'$, $f(i|b') \leq f(i|b)$, where $f(i|b) = f(b \cup$

$\{i\}) - f(b)$.
(ii) $\forall b \in B$ and $i, j, k \in M \setminus b$, $f(i,j|b) + f(k|b) \leq \max\{f(i|b) + f(j,k|b),\ f(j|b) + f(i,k|b)\}$.

We note that the trip value function $V_r^z(b)$ defined on the feasible agent group set B in (3) does not satisfy the monotonicity condition because the size of the combined group $b \cup b'$ may exceed the capacity limit A and the value $V_r^z(b \cup b')$ may be less than $V_r^z(b)$ when the trip sharing disutility is sufficiently high. We denote all agent groups (with sizes both within the vehicle capacity A or larger than A) as $\bar{B} \triangleq 2^M$. Then, we define augmented trip set $\overline{Trip}$ to include trips with any agent group in $\bar{B}$, and define the augmented value function $\bar{V} : \overline{Trip} \to \mathbb{R}$, where $\bar{V}_r^z(\bar{b})$ takes the maximum value of a feasible trip $V_r^z(b)$ with agent group $b \subseteq \bar{b}$. We denote the feasible agent group in $\bar{b}$ that achieves this maximum value as the representative agent group $h_r^z(\bar{b})$:

$$\bar{V}_r^z(\bar{b}) \triangleq \max_{b \subseteq \bar{b},\ b \in B} V_r^z(b), \quad h_r^z(\bar{b}) \triangleq \arg\max_{b \subseteq \bar{b},\ b \in B} V_r^z(b), \quad \forall (\bar{b}, z, r) \in \overline{Trip}. \tag{16}$$

The augmented value function $\bar{V}$ satisfies the monotonicity condition. When all agents have homogeneous trip sharing disutilities, $\bar{V}$ also satisfies the gross substitutes condition. [1]

Lemma 2. *For any $r \in R$ and any $z = 1, \ldots, T - d_r$, the augmented value function $\bar{V}_r^z$ is monotone. Additionally, $\bar{V}_r^z$ satisfies the gross substitutes condition for all $r \in R$ and all z if agents have homogeneous trip sharing disutilities.*

By replacing the original trip value function V with the augmented value function $\bar{V}$ in (LPk^*), we show that the corresponding linear program has an optimal integer solution $\bar{x}^*$ when $\bar{V}$ satisfies the monotonicity and gross substitutes conditions. Furthermore, we show that we can construct an integer optimal solution x^* of the original (LPk^*) by replacing the augmented agent group with the represented agent group in all organized trips in $\bar{x}^*$.

[1] We show that when agents have heterogeneous trip sharing disutilities, the augmented value function may not be gross substitutes. Consider three agents $m = 1, 2, 3$ and a single route r with $d_r = 10$. The latest arrival time for all agents is $T = 11$, and the delay cost is infinity. Thus, we only consider trips with departure time at $z = 1$. The maximum capacity of vehicle is $A = 2$. The preference parameters of agents are $\alpha^1 = \alpha^2 = \alpha^3 = 100$, $\beta^1 = \beta^2 = 6$, $\beta^3 = 4$, $\pi^1(2) = \pi^2(2) = \pi^3(2) = 0$, $\gamma^1(2) = \gamma^2(2) = 0$, and $\gamma^3(2) = 3$. That is, the trip sharing disutilities are heterogeneous. We compute the value function of trips as $V_r^1(\{1\}) = V_r^1(\{2\}) = 40$, $V_r^1(\{3\}) = 70$, $V_r^1(\{1,2\}) = 80$, $V_r^1(\{1,3\}) = V_r^1(\{2,3\}) = 70$. The augmented trip value function is given by $\bar{V}_r^1(\bar{b}) = V_r(b)$ for any $|\bar{b}| \leq 2$, and $\bar{V}_r^1(\{1,2,3\}) = V_r^1(\{1,2\}) = 80$. We can check that $\bar{V}_r^1(\{1\}) + \bar{V}_r^1(\{2,3\}) = 110$, $\bar{V}_r^1(\{2\}) + \bar{V}_r^1(\{1,3\}) = 110$, and $\bar{V}_r^1(\{3\}) + \bar{V}_r^1(\{1,2\}) = 150$. The gross substitutes condition (ii) is violated because $\bar{V}_r^1(\{3\}) + \bar{V}_r^1(\{1,2\}) > \max\left\{\bar{V}_r^1(\{1\}) + \bar{V}_r^1(\{2,3\}), \bar{V}_r^1(\{2\}) + \bar{V}_r^1(\{1,3\})\right\}$. We will show in Section 4 that gross substitutes condition is not only crucial for the equilibrium existence, but also important to guarantee that equilibrium can be computed in polynomial time.

Lemma 3. *The following linear program has an optimal integer solution* $\bar{x}^* = \left(\bar{x}_r^{z*}(\bar{b})\right)_{(\bar{b},z,r)\in\overline{Trip}}$ *if the augmented value function* $\bar{V}$ *satisfies monotonicity and gross substitutes:*

$$\max_{\bar{x}} \quad S(\bar{x}) = \sum_{(\bar{b},z,r)\in\overline{Trip}} \bar{V}_r^z(\bar{b})\bar{x}_r^z(\bar{b}),$$

$$s.t. \quad \sum_{(\bar{b},z,r)\in\{\overline{Trip}|\bar{b}\ni m\}} \bar{x}_r^z(\bar{b}) \le 1, \quad \forall m \in M, \tag{$\overline{\mathrm{LP}}k^*$.a}$$

$$\sum_{\bar{b}\in\bar{B}} \bar{x}_r^z(\bar{b}) \le k_r^{z*}, \quad \forall r \in R, \quad \forall z = 1,\ldots,T-d_r, \tag{$\overline{\mathrm{LP}}k^*$.b}$$

$$\bar{x}_r^z(\bar{b}) \ge 0, \quad \forall \bar{b} \in \bar{B}, \quad \forall (\bar{b},z,r) \in \overline{Trip}. \tag{$\overline{\mathrm{LP}}k^*$.c}$$

Furthermore, given an optimal integer solution $\bar{x}^*$*, any* x^* *that satisfies the following constraints is an optimal integer solution of* (LPk^*)*:*

$$\sum_{b\in h_r^z(\bar{b})} x_r^{z*}(b) = \bar{x}_r^{z*}(\bar{b}), \quad x_r^{z*}(b) \in \{0,1\}, \quad \forall (b,z,r) \in Trip, \quad \forall (\bar{b},z,r) \in \overline{Trip}. \tag{18a}$$

To prove that (LPk^*) has an integer optimal solution, we view each unit capacity of departing at time t and taking route r as a "slot". Thus, given k^*, there are $|L_r^z| = k_r^{z*} = w_r^*$ number of slots for each $r \in R^*$ and each $z = 1,2,\ldots,T-d_r$. The total number of slots is $|L| = \sum_{r\in R} w_r^* \cdot (T - d_r)$. We demonstrate that the agent assignment problem is equivalent to the good allocation problem in an auxiliary economy, where agents are indivisible goods and slots are buyers. Following a similar primal and dual analysis as in Proposition 1, we show that the existence of an integer solution in ($\overline{\mathrm{LP}}k^*$) is equivalent to the existence of Walrasian equilibrium (Kelso Jr and Crawford [1982], see Definition 5 in Appendix C) of our constructed economy. With monotonicity and gross substitutes conditions satisfied, the Walrasian equilibrium exists, and ($\overline{\mathrm{LP}}k^*$) has an integer optimal solution $\bar{x}^*$. Consequently, x^* in (18) is an integer optimal solution of (LPk^*) and, by Lemma 1, also of (LP), concluding the proof of Theorem 1.

3.2 Equivalence to VCG mechanism

In this section, we identify a particular market equilibrium $(x^*, u^\dagger, \tau^\dagger)$ that induces the same outcome as the classical Vickery-Clark-Grove (VCG) mechanism. We show that the $u^\dagger$ achieves the maximum utility for all agents and $\tau^\dagger$ charges the minimum total edge prices among the set of equilibrium (u^*, τ^*). Throughout this section, we assume that the network is series-parallel and agents have homogeneous sharing disutilities, and thus market equilibrium exists following Theorem 1.

A Vickery-Clark-Grove (VCG) mechanism is defined as $(x^*, p^\dagger)$, where x^* is a socially optimal trip organization vector, and payment $p_m^\dagger$ of each agent $m \in M$ is the difference of the total trip values for all other agents given the socially optimal trip organization with and without agent m:

$$p_m^\dagger = S_{-m}(x_{-m}^*) - S_{-m}(x^*), \quad \forall m \in M, \tag{19}$$

where x_{-m}^* is the optimal trip vector with agent set $M \setminus \{m\}$. The optimal social welfare with x_{-m}^* is $S_{-m}(x_{-m}^*)$ given by (IP), and $S_{-m}(x^*) = S(x^*) - \sum_{(z,r,b) \in \{Trip|b \ni m\}} v_{m,r}^z(b) x_r^{z*}(b)$ is the social welfare for agents $M \setminus \{m\}$ with the original optimal trip vector x^*. Given x^* and $p^\dagger$, the utility of each agent $m \in M$ is the difference of the optimal social welfare with and without m:

$$u_m^\dagger \overset{(5)}{=} \sum_{(z,r,b) \in \{Trip|b \ni m\}} v_{m,r}^z(b) x_r^{z*}(b) - p_m^\dagger \overset{(19)}{=} S(x^*) - S_{-m}(x_{-m}^*), \quad \forall m \in M. \tag{20}$$

From the classical theory of mechanism design Ausubel et al. [2006], we know that a VCG mechanism is strategyproof. That means, if there exists a market platform that centrally organizes trips based on agents' reported preference parameters, then given the socially optimal trip organization x^*, and the VCG payment $p^\dagger$, all agents will truthfully report their preferences to the platform.

To show that there exists a strategyproof market equilibrium, it suffices to demonstrate that we can find a price vector $\tau^\dagger$ such that $\left(x^*, p^\dagger, \tau^\dagger\right)$ is a market equilibrium. Next, we show that such $\tau^\dagger$ exists. Moreover, all agents' equilibrium utilities given by $u^\dagger$ are the highest of all market equilibrium, and the total collected edge prices is the minimum.

Theorem 2. *If the network is series-parallel, and agents have homogeneous disutilities of trip sharing, then a strategyproof market equilibrium* $\left(x^*, p^\dagger, \tau^\dagger\right)$ *exists, and the equilibrium utility vector is* $u^\dagger$*. Moreover, given any other market equilibrium* (x^*, p^*, τ^*)*,*

$$u_m^\dagger \geq u_m^*, \quad \forall m \in M, \quad \textit{and} \quad \sum_{t=1}^{T} \sum_{e \in E} q_e \tau_e^{t\dagger} \leq \sum_{t=1}^{T} \sum_{e \in E} q_e \tau_e^{t*}.$$

Theorem 2 shows that there exists a market equilibrium that can be implemented by platforms in a centralized manner – agents report their private preference parameters to the platform, and the platform mediates the market on the agents' behalf. Our result shows that the platform has to implement the equilibrium that maximizes agents' utilities in order to ensure that agents will not lie about their preferences.

We prove Theorem 2 in two steps: Firstly, we show that a utility vector u^* is an equilibrium utility (i.e. there exists a price vector τ^* such that (u^*, τ^*) is an optimal solution of (D)) if and only if there exists a vector $\lambda^* = (\lambda_r^{z*})_{r \in R, z=1,\dots,T}$ such that (u^*, λ^*) is an optimal solution of (Dk^*) – the dual of (LPk^*) (Lemma 9).

$$\begin{aligned}
\min_{u,\lambda} \quad & \sum_{m \in M} u_m + \sum_{r \in R} \sum_{z=1}^{T-d_r} k_r^{z*} \lambda_r^z, \\
s.t. \quad & \sum_{m \in b} u_m + \lambda_r^z \geq V_r^z(b), \quad \forall (b, r, z) \in Trip, && \text{(D}k^*\text{.a)} \\
& u_m \geq 0, \ \lambda_r^z \geq 0, \quad \forall m \in M, \quad \forall z = 1, \dots T - d_r, \quad \forall r \in R. && \text{(D}k^*\text{.b)}
\end{aligned}$$

Here, λ is the dual variable of constraint (LPk^*.b), which can be viewed as the time-dependent price for routes (instead of for edges as in τ). In particular, λ_r^z is the price for any vehicle that departs at z and takes route r. Thus, step 1 indicates that the set of agents' equilibrium utilities with edge-based pricing in the original network is the same as the set of equilibrium utilities achieved with route-based pricing when trips are organized according to the dynamic flow capacity k^*. Secondly, we demonstrate that $u^\dagger$ is an optimal solution of (Dk^*), and the set of all equilibrium utility vectors is a lattice with the maximum element being $u^\dagger$ (Lemma 10). This step leverages the connection between the equilibrium group formation given k^* and the Walrasian equilibrium of the auxiliary economy.

4 Computing Market Equilibrium

We present a polynomial-time algorithm for computing market equilibrium.

Computing optimal trip organization. We compute the optimal trip vector x^* in two steps following Theorem 1: *(Step 1)* Compute the optimal static route capacity vector w^* from Algorithm 1, and compute the dynamic route flow capacity vector k^* as in (14). *(Step 2)* Compute x^* as an optimal integer solution of (LPk^*) by allocating agents to the set of slots L given by k^*. This is done using a modified Kelso-Crawford algorithm (Kelso Jr and Crawford [1982]) for computing Walrasian equilibrium in an equivalent economy with augmented trip value functions $\bar{V}$. Algorithm 2 presents a sketch of step 2, omitting the computation of augmented trip value function $\bar{V}$ and the representative agent group $h_r^z(\bar{b}_l)$. For a complete algorithm and an iterative approach to compute $\bar{V}$ and $h_r^z(\cdot)$ with time complexity of $O(M)$, see Algorithm 3 in Appendix A.

ALGORITHM 2: Allocating agent groups (sketch)

Initialize: Set $u_m \leftarrow 0\ \forall m \in M$; $\bar{b}_l \leftarrow \emptyset,\ \forall l \in L$;

while *TRUE* **do**

    #Compute

    $J_l = \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(J|\bar{b}_l) \triangleq \left\{\bar{V}_l(J \cup \bar{b}_l) - \sum_{m \in \bar{b}_l} u_m - \sum_{m \in J}(u_m + \epsilon)\right\}$ for each $l \in L$;

    for $l \in L$ **do**

        Set $J_l \leftarrow \emptyset$, # Next, we add $m \in M \setminus \bar{b}_l$ to J_l by greedy approach;

        while *True* **do**

            $\hat{m} \leftarrow \arg\max_{m \in M \setminus \{\bar{b}_l \cup J_l\}} \bar{V}_l(\bar{b}_l \cup J_l \cup \{m\}) - \bar{V}_l(\bar{b}_l \cup J_l) - u_m - \epsilon$;

            if $\bar{V}_l(\bar{b}_l \cup J_l \cup \{\hat{m}\}) - \bar{V}_l(\bar{b}_l \cup J_l) - u_{\hat{m}} - \epsilon > 0$ **then**

                $J_l \leftarrow J_l \cup \{\hat{m}\}$;

            else

                break

    # Re-assign agents;

    if $J_l = \emptyset,\ \forall l \in L$ **then**

        break

    else

        Arbitrarily pick $\hat{l}$ with $J_{\hat{l}} \neq \emptyset$, $\bar{b}_{\hat{l}} \leftarrow \bar{b}_{\hat{l}} \cup J_{\hat{l}}$, $\bar{b}_{\hat{l}} \leftarrow \bar{b}_{\hat{l}} \setminus J_{\hat{l}},\ \forall l \neq \hat{l}$, $u_m \leftarrow u_m + \epsilon,\ \forall m \in J_{\hat{l}}$.

return $\left(\bar{b}_l\right)_{l \in L}$

Algorithm 2 starts with setting the utility of all agents to be zero, and the set of agents assigned to each slot $l \in L$ to be empty (Line 1). The algorithm keeps track of (i) each agent m's utility u_m; (ii) the augmented agent group $\bar{b}_l$ that is assigned to each slot $l \in L$; (iii) $\phi_l(\bar{b}_l) = \bar{V}_l(\bar{b}_l) - \sum_{m \in \bar{b}_l} u_m$, which is the difference between the augmented trip value with assigned agent group $\bar{b}_l$ in slot l and the total utility of agents in $\bar{b}_l$, and $J_l = \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(J|\bar{b}_l)$ which is the set of agents, when added to $\bar{b}_l$, maximally increases the value of ϕ_l.

In each iteration of Algorithm 3, Lines 4-14 compute the set J_l based on the current agent group assignment $\bar{b}_l$ and the utility vector u for each $l \in L$. Since the augmented value function $\bar{V}_l(\bar{b})$ satisfies monotonicity and gross substitutes conditions (Lemma 2), we compute the set $J_l = \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(\bar{b}_l \cup J) - \phi_l(\bar{b}_l)$ by iteratively adding agents not in $\bar{b}_l$ into J_l greedily according to their marginal contribution to the value of the function $\phi_l(\bar{b}_l)$ (Gul and Stacchetti [1999], also included in Appendix B). We note that the greedy approach can be used to compute J_l if and only if $\bar{V}_l$ satisfies the monotonicity and gross substitutes conditions (Gul and Stacchetti [1999]). Therefore, the homogeneous trip sharing disutility condition is essential for the polynomial time complexity of the algorithm.

In Lines 16-23, if there exists at least one slot $\hat{l} \in L$ with $J_{\hat{l}} \neq \emptyset$, then we choose one such slot $\hat{l}$, and re-assign agents in group $J_{\hat{l}}$ to the current assignment $\hat{l}$. We increase the utility u_m for the re-assigned agents $m \in J_{\hat{l}}$ by a small number $\epsilon > 0$. The algorithm terminates when $J_l = \emptyset$ for all $l \in L$. The algorithm returns the assigned augmented agent group $\bar{b}_l$ for all $l \in L$ (Line 25). Given $\bar{b}_l$, we compute the representative agent group $b_l = h_r^z(\bar{b}_l)$ for all $l \in L_r^z$ as in (16), see Algorithm 3 in Appendix A for details. Then, x^* is given by $x_r^{z*}(b) = 1$ for all $b \in \{b_l\}_{l \in L_r^z}$, $z \in 1, \ldots, T - d_r$ and $r \in R^*$, and $x_r^{z*}(b) = 0$ for the remaining (z, r, b).

Proposition 2. *For any $\epsilon < \frac{1}{2|M|}$, under the conditions that the network is series-parallel and agents have homogeneous disutilities of trip sharing, the trip organization vector x^* computed by Algorithms 1, 2 is an optimal integer solution of* (IP)*. Moreover, the time complexity of Algorithm 1 is $O(|E||N|^2)$, and the time complexity of Algorithm 2 is $O\left(\frac{V_{max}}{\epsilon}|M|^2|L|\right)$.*

Computing equilibrium payments and edge prices. Given the optimal trip vector x^*, we compute the set of agent payments p^* and edge prices τ^* such that (x^*, p^*, τ^*) is a market equilibrium. Recall from Proposition 1, the equilibrium utilities and edge prices (u^*, τ^*) are optimal solutions of the dual program (D). We can use the Ellipsoid method to compute (u^*, τ^*) given that the separation problem – identifying a violated constraint in (D) for any (u, τ) – can be solved in polynomial time (Nisan and Segal [2006], Grötschel et al. [1993]). To verify constraints (D.a), we need to check whether or not $\max_{b \in B}\{V_r^z(b) - \sum_{m \in b} u_m\} \leq \sum_{e \in r} \tau_e^{z+d_{r,e}}$ is satisfied for all $r \in R$ and $z = 1, \ldots, T - d_r$. We note that $\max_{b \in B}\{V_r^z(b) - \sum_{m \in b} u_m\} = \max_{\bar{b} \in \bar{B}}\{\bar{V}_r^z(\bar{b}) - \sum_{m \in \bar{b}} u_m\}$. Under the monotonicity and gross substitutes conditions, $\max_{\bar{b} \in \bar{B}}\{\bar{V}_r^z(\bar{b}) - \sum_{m \in \bar{b}} u_m\}$ can be computed by greedily adding agents to the set $\bar{b}$ as in Algorithm 2 Lines 5 - 14 (Algorithm 3 Line 3-29). Thus, constraints (D.a) can be verified in $O(|M||R|T)$. Additionally, constraints (D.b) are straightforward to verify. Thus, an equilibrium utility vector u^* and edge price vector τ^* can be computed by the ellipsoid method in time polynomial in $|M|$, $|R|$ and T. Based on x^* and (u^*, τ^*), we can compute the payment vector p^* using (12).

5 Extensions to general network with multiple agent populations

In this section, we generalize the equilibrium existence and computation results to general networks with multiple origin-destination pairs and agents with heterogeneous disutilities of trip sharing. In this generalized setting, the set of all agents M is partitioned into a finite number of subsets $\{M_i\}_{i \in I}$, where agents in different subsets are associated with different origin destination pairs and trip sharing disutilities. For each $i \in I$, we denote the set of routes connecting the origin and destination as R_i. Theorem 1 and Examples 1 – 2 demonstrate that market equilibrium may not exist in the general setting.

To overcome this issue, we consider (i) creating separate market for each agent subset M_i, i.e. agents in each subset M_i only share trips with others in the same subset, and the set of feasible trip groups of market i is B_i; (ii) setting route-based pricing instead of edge-based pricing. That is, the price for any trip in market i to use route r with departure time z is $\lambda^z_{r,i} \geq 0$. We note that edge-based pricing is a special case of route-based pricing since given any edge price vector τ, we can equivalently obtain a route-based price vector where $\lambda^{i,z}_r = \sum_{e \in r} \tau_e^{i,z+d_{r,e}}$. The converse is not necessarily true in that a route-based price vector λ may not correspond to the additive sum of edge-based prices. Following Theorem 1, we can show that market equilibrium (x^*, p^*, λ^*) exists given any capacity allocation vector $q = (q^z_{r,i})_{r \in R_i, i \in I, z=1,\dots,T}$, where $q^z_{r,i}$ is the capacity allocated to market i on route r with departure time z.

How to compute the optimal capacity allocation vector q so that the induced market equilibrium maximize the social welfare? Building on Proposition 1, we formulate the following integer optimization problem:

$$\begin{aligned}
\max \quad & S(x) = \sum_{(z,r,b) \in Trip} V^z_r(b) x^z_r(b) & & \\
s.t. \quad & \sum_{(z,r,b) \in \{Trip | b \ni m\}} x^z_r(b) \leq 1, & \forall i \in I,\ \forall m \in M_i, & \quad (\mathrm{IP_{mult}}.\mathrm{a}) \\
& \sum_{b \in B_i} x^z_r(b) \leq q^{i,z}_r, & \forall i \in I,\ \forall r \in R_i,\ \forall z, & \quad (\mathrm{IP_{mult}}.\mathrm{b}) \\
& \sum_{i \in I} \sum_{r \in \{R_i | r \ni e\}} q^{i,z-d_{r,e}}_r \leq q_e, & \forall e \in E,\ \forall z, & \quad (\mathrm{IP_{mult}}.\mathrm{c}) \\
& x^z_r(b) \in \{0,1\}, \quad q^{i,z}_r \in \mathbb{Z}_+ & \forall i \in I,\ \forall b \in B_i,\ \forall r \in R_i,\ \forall z. & \quad (\mathrm{IP_{mult}}.\mathrm{d})
\end{aligned}$$

Proposition 3. *The problem of computing the socially optimal (q^*, x^*) is NP-hard even if the network is series-parallel.*

The proof of this proposition follows from a reduction of the NP-hard *edge-disjoint paths* problem. This result indicates that exact polynomial-time algorithms for computing the optimal capacity allocation does not exist even if the network is series parallel. The following proposition provides a bound on the *integrality gap* of $(\mathrm{IP_{mult}})$, which is defined as the worst-case ratio of the optimal value of the LP relaxation of $(\mathrm{IP_{mult}})$ to the optimal value of $(\mathrm{IP_{mult}})$.

Proposition 4. *For a general network, the integrality gap of* $(\text{IP}_{\text{mult}})$ *is at least* $\Omega(\max\{k, \sqrt{|E|}\})$. *Moreover, on series-parallel networks, the integrality gap of* $(\text{IP}_{\text{mult}})$ *is at least* $\frac{3}{2}$ *as the time horizon* T *goes to infinity.*

The proof of Proposition 4 exploits the relation between our trip organization problem with the problem of multicommodity integral flow problem. In particular, when group size is restricted to be 1, our problem reduces to the multicommodity flow over time problem. The paper Garg et al. [1993] provided the lower bound on the integrality gap of the multicommodity flow problem in static setting by considering an instance where the underlying network is a grid. We extend their result to show that the same integrality gap holds in our setting, and our proof addresses the subtleties arising due to the temporal nature of the trip organization problem. Moreover, we construct a new problem instance to prove the integrality gap of 3/2 on series parallel network.

Finally, we develop a *Branch-and-Price algorithm* to compute the equilibrium. Let $(\text{LP}_{\text{mult}})$ be the LP-relaxation of $(\text{IP}_{\text{mult}})$, and let (x^*, q^*) be an optimal LP-solution. We note that the computation of the LP relaxation (which has exponential number of variables) builds on the fact that the trip value function in each sub-market satisfies gross substitute condition due to the identical trip sharing disutilities, and thus can be solved efficiently by column generation. If the capacity allocation vector q^* is integral, one can efficiently compute an equilibrium for each submarket via Algorithm 2. Otherwise, there must exist at least one (i, r, z) such that $q_r^{*i,z}$ is fractional. We then branch on this variable to create two subproblems by including one of the two new constraints $q_r^{i,z} \leq \lfloor q_r^{*i,z} \rfloor$, $q_r^{i,z} \geq \lfloor q_r^{*i,z} \rfloor$, and compute the new optimal solution associated with the LP relaxation given the added constraint. The algorithm terminates when all $q_r^{*i,z}$ variables are integer-valued. We include formal description of Algorithm 4 and the implementation details in Appendix F.

6 Application for carpooling and toll pricing in San Francisco Bay Area

We apply our algorithm to the problem of designing the optimal carpooling and toll pricing for the highway network in the San Francisco Bay Area. In this problem, the edge price is the toll price of using that highway segment in a time period, and a shared trip is a carpool trip, where a group of travelers b decides the departure time z and route r that connects their origin and destination. This section provides a brief overview

of the problem instance and key characteristics of the market equilibrium; a detailed report of parameters and computational results are included in Appendix F.

Network. We consider a network with six cities San Francisco, Oakland, San Leandro, Hayward, San Mateo, and South San Francisco, and major highways that connect them (Figure 2). San Francisco city (SF) is the common destination, and the remaining five nodes are the origins. We calibrate the capacity of each edge based on the traffic flow data collected from the highway sensors provided by the California Department of Transportation (https://pems.dot.ca.gov/). We consider each time interval to be 5 minutes, and the entire time period is $T = 60$ minutes between 8am and 9am on workdays.

Populations. For each origin-destination pair (o_i, SF), where o_i is a city in Oakland, San Leandro, Hayward, San Mateo, and South San Francisco, agents traveling from o_i to SF are divided into three populations, each with high (H), medium (M) and low (L) value of time, and disutilities of trip sharing. We estimate the number of agents in each population and their preference parameters based on the driving commuter population size, and their income distributions using the data collected from Safegraph (https://www.safegraph.com/), and US Census of Bureau (https://www.census.gov/), see Appendix F for detailed discussion on parameter estimation.

Results. We compute the optimal capacity allocation, the equilibrium carpool sizes, payments and toll prices using Algorithm 4. We summarize our observations below:

1. Carpooling sizes: In the optimal solution, the L populations from all origins form carpools of size 3 or 4, the M populations form carpools of size 1 or 2, and the H populations do not carpool. We demonstrate the carpool size distribution in Fig. 6b in Appendix F.

2. Payments: We compute the equilibrium payment vector p^* as in (12). The average payment-per-agent for the L populations is \$2, for the M populations is \$4, and for the H populations is around \$5. We demonstrate the payment distribution in Fig. 6c in Appendix F.

3. Tolls: The median toll per route is \$3 for the L populations, \$4 for M populations, and \$8 for the H populations. We plot the dynamic toll prices of all routes in Figure 3.

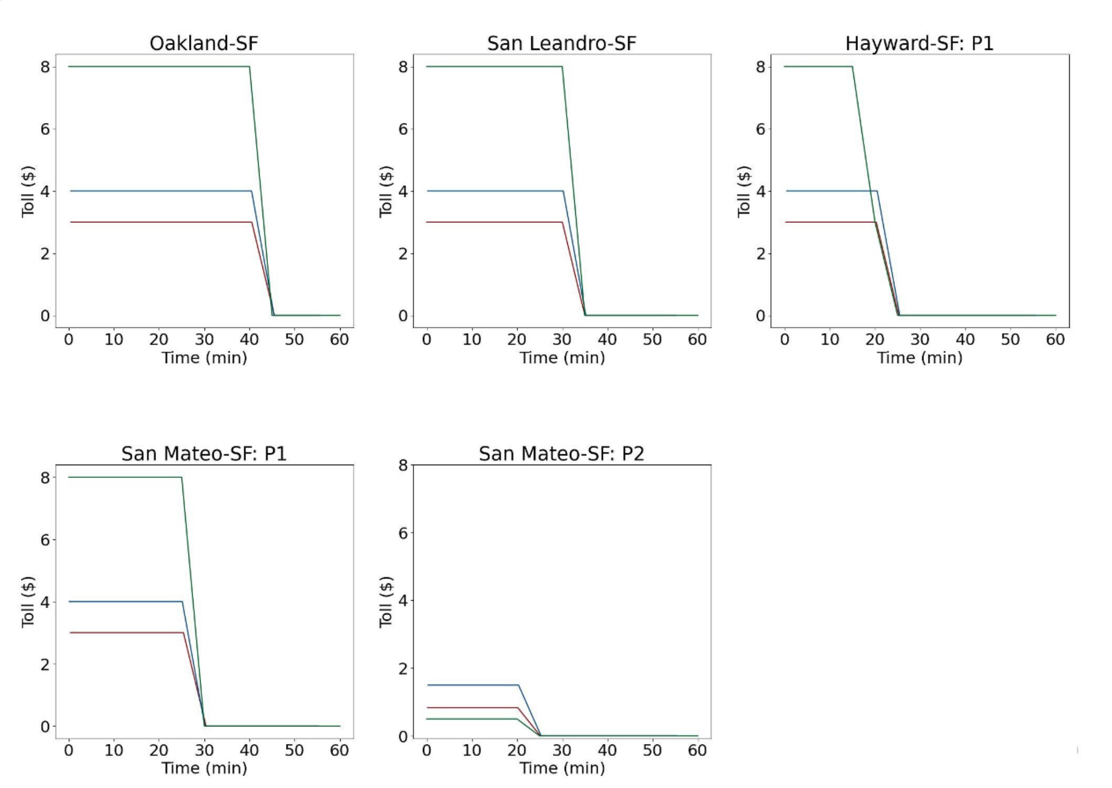

Figure 3: Dynamic toll prices of all routes for H, M, L and populations illustrated in green, blue, and red lines respectively.

7 Concluding Remarks

Our paper focuses on developing a dynamic market mechanism that incentivizes strategic agents to optimize the use of transportation network capacities through group formation and cost sharing. By introducing dynamic edge pricing as the incentive of resource pooling, the paper aims to achieve equilibrium conditions that ensure individual rationality, stability, budget balance, and market clearing. Our results include conditions for equilibrium existence and algorithms for computing market equilibrium. We also extend these findings to general networks with multiple origin-destination pairs, and applying the approach to the San Francisco Bay area highway network as a practical illustration. The results in this paper demonstrates the potential for resource pooling and capacity-based pricing to enhance efficiency, reduce costs, and mitigate environmental impacts in transportation and logistics networks.

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A Agent allocation algorithm

For each each $r \in R^*$, and each $z = 1, \ldots, T - d_r$, we re-write the augmented trip value function (16) with agent group $\bar{b} \in \bar{B}$ and slot $l \in L_r^z$ (with slight abuse of notation) as follows:

$$\bar{V}_l(\bar{b}) = \sum_{m \in h_l(\bar{b})} \eta_{m,l} - \xi_l(|h_l(\bar{b})|), \quad \forall \bar{b} \in \bar{B}, \quad \forall l \in L_r^z, \tag{23}$$

where

$$\eta_{m,l} \triangleq \alpha_m - \beta_m d_r - \ell_m((z + d_r - q_m)_+),$$
$$\xi_l(|h_l(\bar{b})|) \triangleq \big(\pi(|h_l(\bar{b})|) + \sigma\big)\, |h_l(\bar{b})| + \big(\gamma(|h_l(\bar{b})|) + \delta\big)\, |h_l(\bar{b})| d_r,$$

and $h_l(\bar{b}) = h_r^z(\bar{b})$ is the representative agent group in $\bar{b}$ in slot $l \in L_r^z$ as in (16).

Algorithm 3 starts with setting the utility of all agents to be zero, and the set of agents assigned to each slot $l \in L$ to be empty (Line 1). The algorithm keeps track of the following quantities:

- u_m is the utility of each agent $m \in M$.
- $\bar{b}_l$ is the augmented agent group that is assigned to each slot $l \in L$.
- $\bar{h}_l = h_l(\bar{b}_l)$ is the representative agent group given $\bar{b}_l$ in slot $l \in L$. $|\bar{h}_l|$ is the size of the representative agent group.
- $\phi_l(\bar{b}_l) = \bar{V}_l(\bar{b}_l) - \sum_{m \in \bar{b}_l} u_m$ is the difference between the augmented trip value function with assigned agent group $\bar{b}_l$ in slot l and the total utility of agents in $\bar{b}_l$.
- $J_l = \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(\bar{b} \cup J) - \phi_l(\bar{b})$ is the set of agents, when added to $\bar{b}_l$, maximally increases the value of ϕ_l.

In each iteration of Algorithm 3, Lines 3-29 compute the representative agent group $\bar{h}_l$, and the set J_l based on the current agent group assignment and the utility vector for each $l \in L$. In particular, the representative agent group $\bar{h}_l$ is computed by selecting agents from the currently assigned augmented agent group $\bar{b}_l$ in decreasing order of $\eta_{l,m}$ in (23), and the last selected agent $\hat{m}$ (i.e. the agent in $\bar{h}_l$ with the minimum value of $\eta_{l,m}$) satisfies $\eta_{l,\hat{m}} \geq \xi_r(|\bar{h}_l|) - \xi_r(|\bar{h}_l| - 1)$. That is, adding agent $\hat{m}$ to the set $\bar{h}_l \setminus \{\hat{m}\}$ increases the trip value, but adding any other agents decrease the trip value, i.e. $\eta_{l,\hat{m}} < \xi_r(|\bar{h}_l| + 1) - \xi_r(|\bar{h}_l|)$ for all $m \in \bar{b}_l \setminus \bar{h}_l$. The value of $\bar{h}_l$, $|\bar{h}_l|$ records the element and size of the representative

ALGORITHM 3: Allocating trip sharing groups (complete)

Initialize: Set $u_m \leftarrow 0\ \forall m \in M$; $\bar{b}_l \leftarrow \emptyset,\ \forall l \in L$;
while *TRUE* **do**
  for l *in* L **do**
    $J_l \leftarrow \emptyset,\ \bar{h}_l \leftarrow \emptyset,\ |\bar{h}_l| \leftarrow 0,\ \lambda_l \leftarrow 0,\ \phi_l \leftarrow 0, \forall l \in L$;
    for $\hat{m}$ *in* $\mathrm{sort}(\bar{b}_l, \mathrm{key} = \eta_{m,l})$ **do**
      if $\eta_{\hat{m},l} < \big(\xi_l(|\bar{h}_l|+1) - \xi_l(|\bar{h}_l|)\big)\, d_l$ **then**
        break
      else
        $|\bar{h}_l| \leftarrow |\bar{h}_l| + 1,\ \bar{h}_l \leftarrow \bar{h}_l \cup \{\hat{m}\},\ \lambda_l \leftarrow \eta_{\hat{m},l}$
    $\phi_l \leftarrow \sum_{m\in\bar{h}_l} \eta_{m,l} - \xi_l(|\bar{h}_l|)d_l - \sum_{m\in\bar{b}_l} u_m$;
    for $\hat{j}$ *in* $\mathrm{sort}(S \setminus \bar{b}_l, \mathrm{key} = \eta_{j,l} - u_j)$ **do**
      if $\eta_{\hat{j},l} \geq \lambda_l \geq (\xi_l(|\bar{h}_l|+1) - \xi_l(|\bar{h}_l|))d_l$ **then**
        $|\bar{h}'_l| \leftarrow |\bar{h}_l| + 1,\ \bar{h}'_l \leftarrow \bar{h}_l \cup \{\hat{j}\}$,
        $\phi'_l \leftarrow \phi_l + \eta_{\hat{j},l} - (\xi_l(|\bar{h}_l|+1) - \xi_l(|\bar{h}_l|))d_l - u_{\hat{j}} - \epsilon,\ \lambda'_l \leftarrow \lambda_l$
      else if $\lambda_l \geq \eta_{\hat{j},l} \geq (\xi_l(|\bar{h}_l|+1) - \xi_l(|\bar{h}_l|))d_l$ **then**
        $|\bar{h}'_l| \leftarrow |\bar{h}_l| + 1,\ \bar{h}'_l \leftarrow \bar{h}_l \cup \{\hat{j}\}, \lambda'_l \leftarrow \eta_{\hat{j},l}$,
        $\phi'_l \leftarrow \phi_l + \eta_{\hat{j},l} - (\xi_l(|\bar{h}_l|+1) - \xi_l(|\bar{h}_l|))d_l - u_{\hat{j}} - \epsilon$
      else if $\eta_{\hat{j},l} \geq \lambda_l$ *and* $(\xi_l(|\bar{h}_l|+1) - \xi_l(|\bar{h}_l|))d_l \geq \lambda_l$ **then**
        $\bar{h}'_l \leftarrow \bar{h}_l \cup \{\hat{j}\} \setminus \{\underline{l}\}, \lambda'_l \leftarrow \eta_{\hat{j},l},\ \phi'_l \leftarrow \phi_l + \eta_{\hat{j},l} - \lambda_l - u_{\hat{j}} - \epsilon$
      else
        $\phi'_l \leftarrow \phi_l - u_{\hat{j}} - \epsilon$
      if $\phi'_l \leq \phi_l$ **then**
        break
      else
        $(\bar{h}_l, |\bar{h}_l|, \phi_l, \lambda_l) \leftarrow (\bar{h}'_l, |\bar{h}'_l|, \phi'_l, \lambda'_l),\ J_l \leftarrow J_l \cup \{\hat{j}\}$
  if $J_l = \emptyset,\ \forall l \in L$ **then**
    break
  else
    Arbitrarily pick $\hat{l}$ with $J_{\hat{l}} \neq \emptyset$;
    $\bar{b}_{\hat{l}} \leftarrow \bar{b}_{\hat{l}} \cup J_{\hat{l}}$;
    $\bar{b}_{\hat{l}} \leftarrow \bar{b}_{\hat{l}} \setminus J_{\hat{l}},\ \forall l \neq \hat{l}$;
    $u_m \leftarrow u_m + \epsilon,\ \forall m \in J_{\hat{l}}$.
Return $(\bar{b}_l)_{l\in L}$, $(\bar{h}_l)_{l\in L}$

agent group in the current round, and λ_l records the value of $\eta_{l,\hat{m}}$. We also compute the value of the function $\phi_l(\bar{b}_l) = \bar{V}_l(\bar{b}_l) - \sum_{m \in \bar{b}_l} u_m$ in Line 12.

Furthermore, since the augmented value function $\bar{V}_l(\bar{b})$ satisfies monotonicity and gross substitutes conditions, we can compute the set $J_l = \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(\bar{b} \cup J) - \phi_l(\bar{b})$ by iteratively adding agents not in $\bar{b}_l$ into J_l greedily according to their marginal contribution to the value of the function $\phi_l(\bar{b}_l)$ (Gul and Stacchetti [1999]). In this step, we do not re-computed the represented agent group or the augmented trip value function every time we add a agent $\hat{j}$ to $\bar{b}_l$. Instead, we only need to compare $\hat{j}$ with λ_l and the marginal change of $\phi_l(|\bar{h}_l|)$ to determine if the representative agent group needs to include $\hat{j}$ (Lines 13-29).

In Lines 30-38, if there exists at least one slot $\hat{l} \in L$ with $J_{\hat{l}} \neq \emptyset$, then we choose one such slot $\hat{l}$, and re-assign agents in group $J_{\hat{l}}$ to the current assignment $\hat{l}$. We increase the utility u_m for the re-assigned agents $m \in J_{\hat{l}}$ by a small number $\epsilon > 0$. The algorithm terminates when $J_l = \emptyset$ for all $l \in L$. The algorithm returns the representative agent group of each slot $\left(\bar{h}_l\right)_{l \in L}$ (Line 39). The corresponding trip organization vector is given by (18).

B Review of Combinatorial Auction Theory

Consider an economy with a finite set of indivisible goods M and a finite set of buyers L. Each buyer $l \in L$ has a valuation function $\bar{V}_l : \bar{B} \to \mathbb{R}$, where each $\bar{b} \in \bar{B} = 2^M$ is a bundle of goods, and $\bar{V}_l(\bar{b})$ is buyer l's valuation of $\bar{b}$. The good allocation vector in this economy is $\bar{x} = (\bar{x}_l(\bar{b}))_{l \in L, \bar{b} \in \bar{B}}$, where $\bar{x}_l(\bar{b}) = 1$ if good bundle $\bar{b}$ is allocated to buyer l and 0 if otherwise.

Equivalence between group formation and good allocation. Our problem of forming trip sharing groups without vehicle capacity constraint can be equivalently viewed as the good allocation problem in the economy with indivisible goods. In particular, the set of riders M is equivalently viewed as the set of goods M. The set of route and departure time slots L is viewed as the set of buyers. Then, the augmented trip value function $\bar{V}_r^z(\bar{b})$ is equivalent to any buyer $l \in L_r^z$'s valuation of good bundle $\bar{b}$. Each rider m's utility is equivalent to the price of good m.

We next define Walrasian equilibrium of the equivalent economy.

Definition 5 (Walrasian equilibrium Kelso Jr and Crawford [1982])**.** A tuple $(\bar{x}^*, u^*)$ is a Walrasian equilibrium if

(i) For any $l \in L$, $\bar{b}_l \in \arg\max_{\bar{b} \in \bar{B}} \bar{V}_l(\bar{b}) - \sum_{m \in \bar{b}_l} u_m$, where $\bar{b}_l$ is the good bundle that is allocated to l given $\bar{x}^*$, i.e. $\bar{x}_l^*(\bar{b}_l) = 1$

(ii) For any good $m \in M$ that is not allocated to any buyer, (i.e. $\sum_{l \in L} \sum_{\bar{b} \ni m} \bar{x}_l^*(\bar{b}) = 0$), the price $u_m^* = 0$.

Lemma 4 (Kelso Jr and Crawford [1982]). *If the augmented value function $\bar{V}_r^z(\bar{b})$ satisfies the monotonicity and gross substitutes conditions for all $r \in R$ and all $z = 1, \ldots, T - d_r$, then Walrasian equilibrium exists in the equivalent economy with indivisible goods.*

Lemma 5 (Gul and Stacchetti [1999]). *If the value function $\bar{V}$ satisfies the monotonicity and gross substitutes conditions, then the set of Walrasian equilibrium prices U^* is a lattice and has a maximum component $u^\dagger = \left(u_m^\dagger\right)_{m \in M}$ as in (20).*

Lemma 6 (Kelso Jr and Crawford [1982]). *Given any price vector u, if the value function $\bar{V}_l$ for any $l \in L$ satisfies the monotonicity and gross substitutes conditions, then $\bar{b}_l \in \arg\max_{\bar{b} \in \bar{B}} \{\bar{V}_l(\bar{b}) - \sum_{m \in \bar{b}} u_m\}$ can be computed by greedy algorithm.*

Lemma 7 (Kelso Jr and Crawford [1982]). *For any $\epsilon < \frac{1}{2|M|}$, if the value function $\bar{V}_l$ satisfies the monotonicity and gross substitutes conditions for all $l \in L$, then $(\bar{b}_l)_{l \in L}$ computed by Algorithm 3 is a Walrasian equilibrium good allocation.*

C Proof of Statements in Section 2.3

Proof of Proposition 1. First, we proof that the four conditions of market equilibrium (x^*, p^*, τ^*) ensures that x^* satisfies the feasibility constraints of the primal (LP), (u^*, τ^*) satisfies the constraints of the dual (D), and (x^*, u^*, τ^*) satisfies the complementary slackness conditions. Here, the vector u^* is the utility vector computed from (5).

(i) Feasibility constraints of (LP): Since x^* is a feasible trip vector, x^* must satisfy the feasibility constraints of (LP).

(ii) Feasibility constraints of (D): From the stability condition (7), individual rationality (6), and the fact that edge prices are non-negative, we know that (u^*, τ^*) satisfies the feasibility constraints of (D).

(iii) Complementary slackness condition with respect to (LP.a): If rider m is not assigned, then (LP.a) is slack with the integer trip assignment x^* for some rider m. The budget balanced condition (8b) shows that $p_m^* = 0$. Since rider m is not in any trip and the payment is zero, the dual variable (i.e. rider m's utility) $u_m^* = 0$. On the other hand, if $u_m^* > 0$, then rider m must be in a trip, and constraint (LP.a) must be tight. Thus, we can conclude that the complementary slackness condition with respect to the primal constraint (LP.a) is satisfied.

(iv) Complementary slackness condition with respect to (LP.b): Since the mechanism is market clearing, edge price τ_e^t is nonzero if and only if the load that enters edge e at time t is below the capacity, i.e. the primal constraint (LP.b) is slack for edge $e \in E$ and t. Therefore, the complementary slackness condition with respect to the primal constraint (LP.b) is satisfied.

(v) Complementary slackness condition with respect to (D.a): From (8a), we know that for any organized trip, the corresponding dual constraint (D.a) is tight. If constraint (D.a) is slack for a trip (b, r), then the budget balance constraint ensures that trip is not organized. Therefore, the complementary slackness condition with respect to the primal constraint (D.a) is satisfied.

We can analogously show that the inverse of (i) – (v) are also true: the feasibility constraints of (LP) and (D), and the complementary slackness conditions ensure that (x^*, p^*, τ^*) is a market equilibrium. Thus, we can conclude that (x^*, p^*, τ^*) is a market equilibrium if and only if (x^*, u^*, τ^*) satisfies the feasibility constraints of (LP) and (D), and the complementary slackness conditions.

From strong duality theory, we know that the equilibrium trip vector x^* must be an optimal integer solution of (LP). Therefore, the existence of market equilibrium is equivalent to the existence of an integer optimal solution of (LP). The optimal trip assignment is an optimal integer solution of (LP), and (u^*, τ^*) is an optimal solution of the dual problem (D). The payment p^* can be computed from (5). $\square$

D Proof of Statements in Section 3.1

Lemma 8 (Ruzika et al. [2011])**.** *On series-parallel networks, the flow k^* maximizes the total flow that arrives on or before t for every $t = 1, 2, \ldots, T$. That is, for any x that satisfies* (LP.a) – (LP.c)*, we have:*

$$\sum_{r \in R} \sum_{z=1}^{t-d_r} \sum_{b \in B} x_r^z(b) \leq \sum_{r \in R} k_r^* \max\{0, t - d_r\}, \quad \forall t = 1, \ldots, T.$$

Proof of Lemma 1. Consider any (fractional) optimal solution of (LP), denoted as $\hat{x}$. For any time step t, we denote $\hat{f}^t(b) = \sum_{r \in R} \hat{x}_r^{t-d_r}(b)$ as the flow of group b that arrives at the destination at time t. We denote $\widehat{F}^t = \sum_{j=1}^{t} \sum_{b \in B} \hat{f}^j(b)$ as the total flows that arrive at the destination on or before time step t. Since $\hat{x}$ is feasible and the network is series-parallel, we know from Lemma 8 that

$$\widehat{F}^t \leq \sum_{r \in R} k_r \cdot \max\{0, t - d_r\}, \quad \forall t = 1, 2, \ldots, T. \tag{24}$$

We denote the set of all groups with positive flow in $\hat{x}$ as $\widehat{B} \triangleq \{\hat{b} \in B | \sum_{t=1}^{T} \hat{f}^t(\hat{b}) > 0\}$. For each $\hat{b} \in \hat{B}$, we re-write the trip value function in (3) as follows:

$$V_r^z(\hat{b}) = w(\hat{b}) - g(\hat{b})d_r - \sum_{m \in b} \ell_m((z + d_r - \theta_m)_+), \quad \forall \left(\hat{b}, r\right) \in \hat{B} \times R, \quad \forall z = 1, \ldots, T - d_r,$$

where $w(\hat{b}) = \sum_{m \in \hat{b}} \left(\alpha_m - \pi_m(|\hat{b}|)\right) - \sigma|\hat{b}|$, and $g(\hat{b}) = \sum_{m \in \hat{b}} \left(\beta_m + \gamma_m(|\hat{b}|)\right) + \delta|\hat{b}|$ is the sensitivity with respect to travel time cost. We denote the number of agent groups in $\hat{B}$ as n, and re-number these agent groups in decreasing order of $g(\hat{b})$, i.e.

$$g(\hat{b}_1) \geq g(\hat{b}_2) \geq \cdots \geq g(\hat{b}_n).$$

We now construct another trip vector x^* by the following procedure:
Initial zero assignment vector $x_r^{z*}(b) \leftarrow 0$ for all $z = 1, 2, \ldots, T$, $r \in R$ and all $b \in B$.
Initial residual capacity of arriving on or before each time t: $\Delta^t = \sum_{r \in R} k_r^* \cdot \max\{0, t - d_r\} - \widehat{F}^t$.
Initial residual capacity of taking route r to arrive at time t: $\Lambda_r^t = k_r^*$.
For $i = 1, \ldots, n$*:*
For $t = 1, \ldots, T$*:* Re-assign the flow $\hat{f}^t(\hat{b}_i) = \sum_{r \in R} \hat{x}_r^{t-d_r}(\hat{b}_i)$ of group $\hat{b}_i$ that arrive at time t.

(a) Determine the assignable arrival time step set: $\hat{T} = \{t' < t | \Delta^{t'} > 0\} \cup \{t\}$.

(b) Determine assignable route set $\hat{R} = \{r \in R | \sum_{t \in \hat{T}} \Lambda_r^t > 0\}$

(c) Assign agent group $\hat{b}_i$ to a trip that takes route $\hat{r}$ and starts at $\hat{z} = \hat{t} - d_{\hat{r}}$, where $(\hat{r}, \hat{t})$ satisfies:

$$\hat{r} = \arg\min_{r \in \hat{R}} \{d_r\}, \quad \hat{t} = \max_t \{t \in \hat{T} | \Lambda_{\hat{r}}^t > 0\}.$$

If $\hat{t} = t$ and $\Lambda_{\hat{r}}^t \geq \hat{f}^t(\hat{b}_i)$, then $x_{\hat{r}}^{\hat{z}*}(\hat{b}_i) = \hat{f}^t(\hat{b}_i)$. Re-calculate $\Lambda_{\hat{r}}^t \leftarrow \Lambda_{\hat{r}}^t - x_{\hat{r}}^{\hat{z}*}(\hat{b}_i)$.
If $\hat{t} = t$ and $\Lambda_{\hat{r}}^t < \hat{f}^t(\hat{b}_i)$, then $x_{\hat{r}}^{\hat{z}*}(\hat{b}_i) = \Lambda_{\hat{r}}^t$. Re-calculate $\Lambda_{\hat{r}}^t \leftarrow 0$. Repeat (a) - (c).
If $\hat{t} < t$ and $\min\{\Lambda_{\hat{r}}^{\hat{t}}, \Delta^{\hat{t}}\} \geq \hat{f}^t(\hat{b}_i)$, then $x_{\hat{r}}^{\hat{z}*}(\hat{b}_i) = \hat{f}^t(\hat{b}_i)$. Re-calculate $\Lambda_{\hat{r}}^{\hat{t}} \leftarrow \Lambda_{\hat{r}}^{\hat{t}} - x_{\hat{r}}^{\hat{z}*}(\hat{b}_i)$, $\Delta^j \leftarrow \Delta^j - x_{\hat{r}}^{\hat{z}*}(\hat{b}_i)$ for $j = \hat{t}, \ldots, t - 1$.

If $\hat{t} < t$ and $\min\{\Lambda_{\hat{r}}^{\hat{t}}, \Delta^{\hat{t}}\} < \hat{f}^t(\hat{b}_i)$, then $x_{\hat{r}}^{\hat{z}*}(\hat{b}_i) = \min\{\Lambda_{\hat{r}}^{\hat{t}}, \Delta^{\hat{t}}\}$. Re-calculate $\Lambda_{\hat{r}}^{\hat{t}} \leftarrow \Lambda_{\hat{r}}^{\hat{t}} - x_{\hat{r}}^{\hat{z}*}(\hat{b}_i)$, $\Delta^j \leftarrow \Delta^j - x_{\hat{r}}^{\hat{z}*}(\hat{b}_i)$ for $j = \hat{t}, \ldots, t-1$. Repeat (a) - (c).

The re-assignment proceeds to re-assign the flow of $\hat{b} \in \hat{B}$ in decreasing order of their sensitivity with respect to the travel time cost. For each $\hat{b}_i$, the procedure re-assigns the flow of $\hat{b}_i$ that arrives at time t in $\hat{x}$ to a time step before t or at t. In particular, the flow $\hat{f}^t(\hat{b}_i)$ can be re-assigned to arrive at a time step $\hat{t} < t$ only if there is positive residual capacity $\Delta^{\hat{t}}$. Additionally, the re-assignment prioritizes to assign $\hat{f}^t(\hat{b}_i)$ to the route with the minimum travel time cost among all routes that have residual capacity. After assigning $\hat{f}^t(\hat{b}_i)$, the residual arrival capacity Δ and the residual route capacity Λ are re-calculated.

We now check that the constructed trip assignment vector is a feasible solution of (LPk^*). Since we only re-assigned trips with positive weight in $\hat{x}$, we know that $\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\ni m} x_r^{z*}(b) \leq 1$, and thus x^* satisfies (LPk^*.a). Additionally, we note that in all steps of assignment, the total flow of trips that use each r and starts at time z is less than the capacity in the temporal repeated flow k_r^*. Thus, we have $\sum_{b\in B} x_r^{z*}(b) \leq k_r^{z*}$ for all $r \in R$ and for all $z \in T$. Therefore, (LPk^*.b) is satisfied. Thus, the constructed x^* is a feasible solution of (LPk^*).

It remains to prove that x^* is optimal of (LPk^*). We prove this by showing that $S(x^*) \geq S(\hat{x})$. The objective function $S(x^*)$ can be written as follows:

$$\sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R} V_r^z(b)x_r^{z*}(b) = \sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R} w(b)x_r^{z*}(b) - \sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R} g(b)d_r x_r^{z*}(b)$$
$$- \sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R}\left(\sum_{m\in b}\ell_m((z+d_r-\theta_m)_+)\right)x_r^{z*}(b). \tag{25}$$

We note that the flow $\hat{f}^t(\hat{b})$ for each t and $\hat{b}$ is re-assigned with the same weight to arrive on or before t since the total flow that arrive on or before t (i.e. $\hat{F}^t$) is no higher than $\sum_{r\in R} k_r^* \max\{0, t-d_r\}$ (i.e. k^* is the earliest arrival flow), and a flow that arrive later than t is re-assigned to arrive before t only if the residual arrival capacity $\Delta^t > 0$. Such re-assignment of arrival time does not occupy capacity for the flow that previously arrive on or before t in $\hat{x}$. As a result, the re-assignment process must terminate with all agent groups being assigned with the same weight as in $\hat{x}$, i.e. $\sum_{r\in R}\sum_{z=1}^{T} x_r^{z*}(b) = \sum_{r\in R}\sum_{z=1}^{T}\sum_{r\in R}\hat{x}_r^z(b)$ for all $b \in B$. Therefore,

$$\sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R} w(b)x_r^{z*}(b) = \sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R} w(b)\hat{x}_r^z(b). \tag{26}$$

Additionally, in the reassignment process (a), we note that all agent groups $\hat{b}_1, \ldots, \hat{b}_n$ with positive weight in $\hat{x}$ are assigned to arrive at a time in x^* that is no later than the arrival time in $\hat{x}$. Therefore, we know that

$$\sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R}\left(\sum_{m\in b}\ell_m((z+d_r-\theta_m)_+)\right)x_r^{z*}(b)\leq\sum_{b\in B}\sum_{z=1}^{T}\sum_{r\in R}\left(\sum_{m\in b}\ell_m((z+d_r-\theta_m)_+)\right)\hat{x}_r^{z}(b).$$

To prove $S(x^*)\geq S(\hat{x}^*)$, we only need to show that

$$\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B}g(b)d_r x_r^{z*}(b)\leq\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B}g(b)d_r\hat{x}_r^{z}(b)$$

To do this, we next show that x^* is an optimal solution of the following problem:

$$\begin{aligned} & x^*\in\arg\min_{x\in X(\hat{f})}\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B}g(b)d_r x_r^z(b), \\ s.t.\quad & X(\hat{f})\triangleq\left\{x\;\middle|\;\begin{array}{l}\sum_{r\in R}\sum_{j=1}^{t}x_r^{j-d_r}(b)\geq\sum_{j=1}^{t}\hat{f}^j(b),\quad\forall b\in B,\quad\forall t=1,\dots,T-1,\\ \sum_{r\in R}\sum_{z=1}^{T}x_r^z(b)=\sum_{j=1}^{T}\hat{f}^j(b),\quad\forall b\in B,\\ \sum_{b\in B}\sum_{r\ni e}x_r^{t-d_{r,e}}(b)\leq q_e,\quad\forall e\in E,\quad\forall t=1,\dots,T,\\ x_r^z(b)\geq 0,\quad\forall r\in R,\quad\forall b\in B,\quad\forall z=1,\dots,T.\end{array}\right\}.\end{aligned}\tag{27}$$

That is, we will show that x^* derived from the re-assignment of flow $\hat{f}$ induced by $\hat{x}$ minimizes the value of $\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B}g(b)d_r x_r^z(b)$ across all trip organization vectors that allocate the same weight of each b as in $\hat{x}$ and do not increase the arrival time of any flow of b. In particular, $X(\hat{f})$ characterizes the set of all such x: the first constraint ensures that the flow of each b given x arrives at the destination at time on or before t is no less than that in $\hat{x}$; The second constraint ensures that the total flow of each b is assigned with the same weight in x as that in $\hat{x}$; The third constraint ensures that x satisfies the edge capacity constraint in all time steps; and the last constraint ensures the non-negativity of x. The intuition of (27) is that in the re-assignment procedure, agent group $\hat{b}$ with higher sensitivity with respect to travel time cost is assigned first, and prioritized to take shorter routes.

We prove (27) by mathematical induction. To begin with, (27) holds trivially on any single-link network since no-reassignment is needed with a single route. We next prove that if (27) holds on two series-parallel sub-networks G^1 and G^2, then (27) holds on the network G that connects G^1 and G^2 in series or in parallel. In particular, we analyze the cases of series connection and parallel connection separately:

(Case 1) Series-parallel network G is formed by connecting two series-parallel sub-networks

G^1 and G^2 in series. We denote the set of routes in subnetwork G^1 and G^2 as R^1 and R^2, respectively. Since G^1 and G^2 are connected in series, the set of routes in network G is $R \triangleq R^1 \times R^2$.

We define the set of feasible trip vectors in the subnetwork G^2 that can induce a flow satisfying the arrival time constraint as follows:

$$X^2(\hat{f}) \triangleq \left\{ x \;\middle|\; \begin{array}{l} \sum_{r^2 \in R^2} \sum_{b \in B} x_{r^2}^{z^2}(b) \leq \sum_{r^1 \in R^1} \max\{0, z^2 - d_{r^1}\} \left(\sum_{r^2 \in R^2} k^*_{r^1 r^2} \right), \ \forall z^2 = 1, \dots, T, \\ \sum_{r^2 \in R^2} \sum_{z^2=1}^{T} x_{r^2}^{z^2}(b) = \sum_{j=1}^{T} \hat{f}^j(b), \quad \forall b \in B, \\ \sum_{r^2 \in R^2} \sum_{j=1}^{t} x_{r^2}^{j-d_{r^2}}(b) \geq \sum_{j=1}^{t} \hat{f}^j(b), \quad \forall b \in B, \quad \forall t = 1, \dots, T-1, \\ \sum_{b \in B} \sum_{r^2 \ni e} x_{r^2}^{t-d_{r^2,e}}(b) \leq q_e, \quad \forall e \in E^2, \quad \forall t = 1, \dots, T, \\ x_{r^2}^{z^2}(b) \geq 0, \quad \forall r^2 \in R^2, \quad \forall b \in B, \quad \forall z^2 = 1, \dots, T. \end{array} \right\},$$

where the first constraint ensures that the flow departing from the origin of G^2 at any z^2 does not exceed the maximum flow that can arrive at the destination of G^1 before z^2.

Given $\hat{x}$ (and the induced flow vector $\hat{f}$), the trip vector obtained from the re-assignment procedure restricted to the subnetwork G^2 is $x^{2*} = (x_{r^2}^{z^2 *}(b))_{r^2 \in R^2, b \in B, z^2 = 1, \dots, T}$, where

$$x_{r^2}^{z^2 *}(b) = \sum_{r^1 \in R^1} x_{r^1 r^2}^{z^2 - d_{r^1} *}(b), \quad \forall r^2 \in R^2, \quad \forall z^2 = 1, \dots, T, \quad \forall b \in B.$$

We can check that $x^{2*} \in X^2(\hat{f})$. Therefore, according to the induction assumption, we have

$$\sum_{r^2 \in R^2} \sum_{z^2=1}^{T} \sum_{b \in B} g(b) d_{r^1} x_{r^2}^{z^2 *}(b) \leq \sum_{r^2 \in R^2} \sum_{z^2=1}^{T} \sum_{b \in B} g(b) d_{r^2} x_{r^2}^{z^2}(b), \quad \forall x^2 \in X^2(\hat{f}). \tag{28}$$

Additionally, given any x^{2*}, the set of feasible trip vector restricted to the subnetwork G^1 is given by

$$X^1(\hat{f}) \triangleq \left\{ x^1 \;\middle|\; \begin{array}{l} \sum_{r^1 \in R^1} \sum_{z^1=1}^{T} x_{r^1}^{z^1}(b) = \sum_{j=1}^{T} \sum_{r^2 \in R^2} x_{r^2}^{j*}(b), \quad \forall b \in B, \\ \sum_{r^1 \in R^1} \sum_{j=1}^{t} x_{r^1}^{j-d_{r^1}}(b) \geq \sum_{j=1}^{t} \sum_{r^2 \in R^2} x_{r^2}^{j*}(b), \quad \forall b \in B, \quad \forall t = 1, \dots, T-1, \\ \sum_{b \in B} \sum_{r^1 \ni e} x_{r^1}^{t-d_{r^1,e}}(b) \leq q_e, \quad \forall e \in E^1, \quad \forall t = 1, \dots, T, \\ x_{r^1}^{z^1}(b) \geq 0, \quad \forall r^1 \in R^1, \quad \forall b \in B, \quad \forall z^1 = 1, \dots, T. \end{array} \right\}$$

We consider $x^{1*} = (x_{r^1}^{z^1 *}(b))_{r^1 \in R^1, b \in B, z^1 = 1, \dots, T}$, where

$$x_{r^1}^{z^1 *}(b) = \sum_{r^2 \in R^2} x_{r^1 r^2}^{z^1 *}(b), \quad \forall r^1 \in R^1, \quad \forall z^1 = 1, \dots, T, \quad \forall b \in B.$$

Analogous to our argument on G^2, we can check that $x^{1*} \in X^1(\hat{f})$. Again from the induction assumption, x^{1*} satisfies

$$\sum_{r^1\in R^1}\sum_{z^1=1}^{T}\sum_{b\in B} g(b)d_{r^1}x_{r^1}^{z^1*}(b) \leq \sum_{r^1\in R^1}\sum_{z^1=1}^{T}\sum_{b\in B} g(b)d_{r^1}x_{r^1}^{z^1}(b), \quad \forall x^1 \in X^1(\hat{f}). \tag{29}$$

From (28) and (29), we obtain that

$$\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_r x_r^z(b) = \sum_{r^1\in R^1}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_{r^1}\left(\sum_{r^2\in R^2} x_{r^1r^2}^z(b)\right) + \sum_{r^2\in R^2}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_{r^2}\left(\sum_{r^1\in R^1} x_{r^1r^2}^z(b)\right)$$
$$= \sum_{r^1\in R^1}\sum_{z^1=1}^{T}\sum_{b\in B} g(b)d_{r^1}x_{r^1}^{z^1}(b) + \sum_{r^2\in R^2}\sum_{z^2=1}^{T}\sum_{b\in B} g(b)d_{r^2}x_{r^2}^{z^2}(b)$$
$$\geq \sum_{r^1\in R^1}\sum_{z^1=1}^{T}\sum_{b\in B} g(b)d_{r^1}x_{r^1}^{z^1*}(b) + \sum_{r^2\in R^2}\sum_{z^2=1}^{T}\sum_{b\in B} g(b)d_{r^2}x_{r^2}^{z^2*}(b) = \sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_r x_r^{z*}(b).$$

Thus, we have proved that (27) holds on G when G^1 and G^2 are connected in series.
(Case 2) Series-parallel Network G is formed by connecting two series-parallel networks G_1 and G_2 in parallel. Same as case 1, we denote R^1 (resp. R^2) as the set of routes in G^1 (resp. G^2). Then, the set of all routes in G is $R = R^1 \cup R^2$.

Given any $\hat{f}$, we compute x^* from the re-assignment procedure in network G. We denote $f^{t,1*}(b) = \sum_{r^1\in R^1} x_{r^1}^{t-d_{r^1}*}(b)$ (resp. $f^{t,2*}(b) = \sum_{r^2\in R^2} x_{r^2}^{t-d_{r^2}*}(b)$) as the total flow of agent group b that arrives at the destination at time t using routes in the subnetwork G^1 (resp. G^2) given the organization vector x^*. We now denote x^{1*} (resp. x^{2*}) as the trip vector x^* restricted on sub-network G^1 (resp. G^2), i.e. $x^{1*} = \left(x_{r^1}^{z*}(b)\right)_{r^1\in R^1, b\in B, z=1,\dots,T}$ (resp. $x^{2*} = \left(x_{r^2}^{z*}(b)\right)_{r^2\in R^2, b\in B, z=1,\dots,T}$). We can check that x^{1*} (resp. x^{2*}) is the trip vector obtained by the re-assignment procedure given the total flow $f^{t,1*}$ (resp. $f^{t,2*}$) on network G^1 (resp. G^2).

Consider any arbitrary split of the total flow $\hat{f}$ to the two sub-networks, denoted as $\left(\hat{f}^1, \hat{f}^2\right)$, such that $\hat{f}^{t,1}(b) + \hat{f}^{t,2}(b) = \hat{f}^t(b)$ for all $b \in B$ and all $t = 1, 2, \dots, T$. Given $\hat{f}^1$ (resp. $\hat{f}^2$), we denote the trip vector obtained by the re-assignment procedure on sub-network G^1 (resp. G^2) as $\hat{x}^{1*}$ (resp. $\hat{x}^{2*}$). We also define the set of feasible trip vectors on sub-network G^1 (resp. G^2) that induce the total flow $\hat{f}^1$ (resp. $\hat{f}^2$) given by (27) as $X^1(\hat{f}^1)$ (resp. $X^2(\hat{f}^2)$). Then, the set of all trip vectors that induce $\hat{f}$ on network G is $X(\hat{f}) = \cup_{\left(\hat{f}^1, \hat{f}^2\right)}(X^1(\hat{f}^1), X^2(\hat{f}^2))$.

Under our assumption that (27) holds on sub-network G^1 and G^2 with any total flow, we know that given any flow split $\left(\hat{f}^1, \hat{f}^2\right)$,

$$\begin{aligned}&\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_{r^1}\hat{x}_{r^1}^{z*}(b) + \sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_{r^2}\hat{x}_{r^2}^{z*}(b)\\ \leq &\sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_{r^1}\hat{x}_{r^1}^{z}(b) + \sum_{r\in R}\sum_{z=1}^{T}\sum_{b\in B} g(b)d_{r^2}\hat{x}_{r^2}^{z}(b), \qquad \forall \hat{x}^1 \in X(\hat{f}^1), \quad \forall \hat{x}^2 \in X(\hat{f}^2).\end{aligned}$$

Therefore, the optimal solution of (27) must be a trip vector $(\hat{x}^{1*}, \hat{x}^{2*})$ associated with a flow split $\left(\hat{f}^1, \hat{f}^2\right)$. It thus remains to prove that any $(\hat{x}^{1*}, \hat{x}^{2*})$ associated with flow split $\left(\hat{f}^1, \hat{f}^2\right) \neq (f^{1*}, f^{2*})$ cannot be an optimal solution (i.e. can be improved by re-arranging flows).

For any $\left(\hat{f}^1, \hat{f}^2\right) \neq (f^{1*}, f^{2*})$, we can find a group b_j and a time t such that $\hat{f}^{t,1}(b_j) \neq f^{t,1*}(b_j)$ (henceforth $\hat{f}^{t,2}(b_j) \neq f^{t,2*}(b_j)$). We denote $b_{\hat{j}}$ as one such group with the maximum $g(b)$, and $\hat{t}$ as minimum of such time step, i.e. $\hat{f}^{t,i}(b_j) = f^{t,i*}(b_j)$ for any $i = 1, 2$, any $j = 1, \ldots, \hat{j} - 1$ and any $t = 1, \ldots, T$. Additionally, $\hat{f}^{t,i}(b_{\hat{j}}) = f^{t,i*}(b_{\hat{j}})$ for $i = 1, 2$, and $t = 1, \ldots \hat{t}-1$. Since groups $b_1, \ldots, b_{\hat{j}-1}$ are assigned before group $b_{\hat{j}}$, we know that $\hat{x}_{r^1}^{z*}(b_j) = x_{r^1}^{z*}(b_j)$ and $\hat{x}_{r^2}^{z*}(b_j) = x_{r^2}^{z*}(b_j)$ for all $r^1 \in R^1$, all $r^2 \in R^2$ and all $j = 1, \ldots, \hat{j} - 1$.

Without loss of generality, we assume that $\hat{f}^{\hat{t},1}(b_{\hat{j}}) > f^{\hat{t},1*}(b_{\hat{j}})$ and $\hat{f}^{\hat{t},2}(b_{\hat{j}}) < f^{\hat{t},2*}(b_{\hat{j}})$. Then, there must exist routes $\hat{r}^1 \in R^1$ and $\hat{r}^2 \in R^2$, and departure time z^1, z^2 such that $d_{\hat{r}^1} + z^1 \leq \hat{t}$, $d_{\hat{r}^2} + z^2 \leq \hat{t}$, $\hat{x}_{\hat{r}^1}^{z^1*}(b_{\hat{j}}) > x_{\hat{r}^1}^{z^1*}(b_{\hat{j}})$ and $\hat{x}_{\hat{r}^2}^{z^2*}(b_{\hat{j}}) < x_{\hat{r}^2}^{z^2*}(b_{\hat{j}})$. Moreover, since x^* assigns group $b_{\hat{j}}$ to routes with the minimum travel time cost that are unsaturated after assigning groups $b_1, \ldots, b_{\hat{j}-1}$ with all arrival time t and group $b_{\hat{j}}$ with arrival time earlier than $\hat{t}$, we have $d_{\hat{r}^2} < d_{\hat{r}^1}$. If the total flow on route $\hat{r}^2$ with departure time z^2 is less than $k_{r^2}^*$ (unsaturated) given $\hat{x}^{2*}$, then we decrease $\hat{x}_{\hat{r}^1}^{z^1*}(b_{\hat{j}})$ and increase $\hat{x}_{\hat{r}^2}^{z^2*}(b_{\hat{j}})$ by a small positive number $\epsilon > 0$. We can check that the objective function of (27) is reduced by $\epsilon(d_{\hat{r}^1} - d_{\hat{r}^2})\epsilon g(b_{\hat{j}}) > 0$. On the other hand, if route $\hat{r}^2$ with departure time z^2 is saturated, then another group $b_{\hat{j}'}$ with $\hat{j}' > \hat{j}$ must be assigned to $\hat{r}^2$ with departure time z^2. Then, we decrease $x_{\hat{r}^1}^{z^1*}(b_{\hat{j}})$ and $x_{\hat{r}^2}^{z^2*}(b_{\hat{j}'})$ by $\epsilon > 0$, increases $x_{\hat{r}^1}^{z^1*}(b_{\hat{j}'})$ and $x_{\hat{r}^2}^{z^2*}(b_{\hat{j}})$ by ϵ (i.e. exchange a small fraction of group $b_{\hat{j}}$ with group $b_{\hat{j}'}$). Note that $g(b_{\hat{j}}) > g(b_{\hat{j}'})$ and $d_{\hat{r}^1} > d_{\hat{r}^2}$. We can thus check that the objective function of (27) is reduced by $\epsilon(d_{\hat{r}^1}g(b_{\hat{j}}) - d_{\hat{r}^2}g(b_{\hat{j}+1}))\epsilon > 0$. Therefore, we have found an adjustment of trip vector $(\hat{x}^{1*}, \hat{x}^{2*})$ that reduces the objective function of (27). Hence, for any flow split $\left(\hat{f}^1, \hat{f}^2\right) \neq (f^{t,1*}, f^{t,2*})$, the associated trip vector $(\hat{x}^{1*}, \hat{x}^{2*})$ is not the optimal solution of (27). The optimal solution of (27) must be constructed by the re-assignment procedure with flow split $(f^{t,1*}, f^{t,2*})$, i.e. must be x^*.

We have shown from cases 1 and 2 that if the solutions derived from the re-assignment procedure minimizes (27) on the two series-parallel sub-networks, then x^* derived from the re-assignment procedure must also minimize (27) on the connected series-parallel network. Moreover, since (27) is minimized trivially when the network is a single edge, and any series-parallel network is formed by connecting series-parallel sub-networks in series or parallel, we can conclude that x^* obtained from the re-assignment procedure minimizes the objective function in (27) for any flow vector $\hat{f}$ on any series-parallel network.

From (25), (26) and (27), we can conclude that $S(x^*) \geq S(\hat{x})$. Since x^* is a feasible solution of (LPk^*), the optimal value of (LPk^*) must be no less than that of (LP). On the other hand, since the constraints in (LP) are less restrictive than that in (LPk^*), the optimal value of (LPk^*) is no higher than that of (LP). Therefore, the optimal value of (LPk^*) equals to that of (LP), and any optimal solution of (LPk^*) must also be an optimal solution of (LP). $\square$

Proof of Lemma 2. The augmented value function satisfies monotonicity condition since for any $\bar{b} \subseteq \bar{b}'$, we have:

$$\bar{V}_r^z(\bar{b}) = \max_{b \subseteq \bar{b},\ b \in B} V_r^z(b) \leq \max_{b \subseteq \bar{b}',\ b \in B} V_r^z(b) = \bar{V}_r^z(\bar{b}').$$

We next prove that $\bar{V}_r^z$ satisfies gross substitutes condition. Since all agents have homogeneous disutility of trip sharing, we can simplify the trip value function $\bar{V}_r^z(\bar{b})$ as follows:

$$\bar{V}_r^z(\bar{b}) = \sum_{m \in h_r^z(\bar{b})} \eta_{m,r}^z - \theta(|h_r^z(\bar{b})|),$$

where

$$\begin{aligned} \eta_{m,r}^z &\triangleq \alpha_m - \beta_m d_r - \ell_m((z + d_r - \theta_m)_+), \\ \xi_r(|h_r^z(\bar{b})|) &\triangleq \left(\pi(|h_r^z(\bar{b})|) + \sigma\right) |h_r^z(\bar{b})| + \left(\gamma(|h_r^z(\bar{b})|) + \delta\right) |h_r^z(\bar{b})| d_r. \end{aligned}$$

Before proving that the augmented trip value function $\bar{V}_r^z(\bar{b})$ satisfies (a) and (b) in Definition 4, we first provide the following statements that will be used later:

(i) The function $\theta(|h_r^z(\bar{b})|)$ is non-decreasing in $|h_r^z(\bar{b})|$ because the marginal disutility of trip sharing is non-decreasing in the group size.

(ii) The representative agent group for any trip can be constructed by selecting agents from $\bar{b}$ in decreasing order of $\eta_{m,r}^z$. The last selected agent $\hat{m}$ (i.e. the agent in $h_r^z(\bar{b})$ with the minimum value of $\eta_{m,r}^z$) satisfies:

$$\eta_{\hat{m},r}^z \geq \theta(|h_r^z(\bar{b})|) - \theta(|h_r^z(\bar{b})| - 1). \tag{30}$$

That is, adding agent $\hat{m}$ to the set $h_r^z(\bar{b}) \setminus \{\hat{m}\}$ increases the trip valuation. Additionally,

$$\eta_{m,r}^z < \theta(|h_r^z(\bar{b})| + 1) - \theta(|h_r^z(\bar{b})|), \quad \forall m \in \bar{b} \setminus h_r^z(\bar{b}). \tag{31}$$

Then, adding any agent in $\bar{b} \setminus h_r^z(\bar{b})$ to $h_r^z(\bar{b})$ no longer increases the trip valuation.

(iii) $|h_r^z(\bar{b}')| \geq |h_r^z(\bar{b})|$ for any two agent groups $\bar{b}', \bar{b} \in B$ such that $\bar{b}' \supseteq \bar{b}$.

Proof of (iii). Assume for the sake of contradiction that $|h_r^z(\bar{b}')| < |h_r^z(\bar{b})|$. Consider the agent $\hat{m} \in \arg\min_{m \in h_r^z(\bar{b})} \eta_{m,r}^z$. The value $\eta_{\hat{m},r}^z$ satisfies (30). Since $|h_r^z(\bar{b}')| < |h_r^z(\bar{b})|$, $\bar{b}' \supseteq \bar{b}$, and we know that agents in the representative agent group $h_r^z(\bar{b}')$ are the ones with $|h_r^z(\bar{b}')|$ highest $\eta_{m,r}^z$ in $\bar{b}'$, we must have $\hat{m} \notin h_r^z(\bar{b}')$. From (31), we know that $\eta_{\hat{m},r}^z < \theta(|h_r^z(\bar{b}')| + 1) - \theta(|h_r^z(\bar{b}')|)$. Since the marginal disutility of trip sharing is non-decreasing in the agent group size, we can check that $\theta(|h_r^z(\bar{b})| + 1) - \theta(|h_r^z(\bar{b})|)$ is non-decreasing in $|h_r^z(\bar{b})|$. Since $|h_r^z(\bar{b}')| < |h_r^z(\bar{b})|$, we have $|h_r^z(\bar{b}')| \leq |h_r^z(\bar{b})| - 1$. Therefore,

$$\eta_{\hat{m},r}^z < \theta(|h_r^z(\bar{b}')| + 1) - \theta(|h_r^z(\bar{b}')|) \leq \theta(|h_r^z(\bar{b})|) - \theta(|h_r^z(\bar{b})| - 1),$$

which contradicts (30) and the fact that $\hat{m} \in h_r^z(\bar{b})$. Hence, $|h_r^z(\bar{b}')| \geq |h_r^z(\bar{b})|$.

We now prove that $\bar{V}_r^z$ satisfies *(i)* in Definition 4. For any $\bar{b}, \bar{b}' \subseteq M$ and $\bar{b} \subseteq \bar{b}'$, consider two cases:

Case 1: $i \notin h_r^z(\{i\} \cup \bar{b}')$. In this case, $h_r^z(\bar{b}' \cup i) = h_r^z(\bar{b}')$, and $\bar{V}_r^z(i|\bar{b}') = \bar{V}_r^z(\bar{b}' \cup i) - \bar{V}_r^z(\bar{b}') = 0$. Since $\bar{V}_r^z$ satisfies monotonicity condition, we have $\bar{V}_r^z(i|\bar{b}) \geq 0$. Therefore, $\bar{V}_r^z(i|\bar{b}) \geq \bar{V}_r^z(i|\bar{b}')$.

Case 2: $i \in h_r^z(\{i\} \cup \bar{b}')$. We argue that $i \in h_r^z(\{i\} \cup \bar{b})$. From (30), $\eta_{i,r}^z \geq \theta(|h_r^z(\bar{b}')|) - \theta(|h_r^z(\bar{b}')| - 1)$. Since $\bar{b}' \supseteq \bar{b}$, we know from (iii) that $|h_r^z(\bar{b}')| \geq |h_r^z(\bar{b})|$. Hence, $\eta_{i,r}^z \geq \theta(|h_r^z(\bar{b})|) - \theta(|h_r^z(\bar{b})| - 1)$, and thus $i \in h_r^z(\{i\} \cup \bar{b})$.

We define $\hat{m}' \triangleq \arg\min_{m \in h_r^z(\bar{b}')} \eta_{m,r}^z$ and $\hat{m} \triangleq \arg\min_{m \in h_r^z(\bar{b})} \eta_{m,r}^z$. We also consider two thresholds $\mu' = \theta(|h_r^z(\bar{b}')| + 1) - \theta(|h_r^z(\bar{b}')|)$, and $\mu = \theta(|h_r^z(\bar{b})| + 1) - \theta(|h_r^z(\bar{b})|)$. Since $\bar{b}' \supseteq \bar{b}$, from (iii), we have $|h_r^z(\bar{b}')| \geq |h_r^z(\bar{b})|$ and thus $\mu' \geq \mu$. We further consider four sub-cases:

(2-1) $\eta_{\hat{m}',r}^z \geq \mu'$ and $\eta_{\hat{m},r}^z \geq \mu$. From (30) and (31), $h_r^z(\{i\} \cup \bar{b}') = h_r^z(\bar{b}') \cup \{i\}$ and $h_r^z(\{i\} \cup \bar{b}) = h_r^z(\bar{b}) \cup \{i\}$. The marginal value of i is $\bar{V}_r^z(i|\bar{b}') = \eta_{i,r}^z - \mu'$, and $\bar{V}_r^z(i|\bar{b}) = \eta_{i,r}^z - \mu$. Since $\mu' \geq \mu$, $\bar{V}_r^z(i|\bar{b}') \leq \bar{V}_r^z(i|\bar{b})$.

(2-2) $\eta_{\hat{m}',r}^z < \mu'$ and $\eta_{\hat{m},r}^z \geq \mu$. Since $i \in h_r^z(\{i\} \cup \bar{b}')$ in *Case 2*, we know from (30) and (31) that $h_r^z(\{i\} \cup \bar{b}') = h_r^z(\bar{b}') \setminus \{\hat{m}'\} \cup \{i\}$ and $h_r^z(\{i\} \cup \bar{b}) = h_r^z(\bar{b}) \cup \{i\}$. Therefore, $\bar{V}_r^z(i|\bar{b}') = \eta_{i,r}^z - \eta_{\hat{m}',r}^z$ and $\bar{V}_r^z(i|\bar{b}) = \eta_{i,r}^z - \mu$. We argue in this case, we must have $|h_r^z(\bar{b}')| > |h_r^z(\bar{b})|$. Assume for the sake of contradiction that $|h_r^z(\bar{b}')| = |h_r^z(\bar{b})|$, then $\mu' = \mu$ and $\eta_{\hat{m}',r}^z \geq \eta_{\hat{m},r}^z$ because $\bar{b}' \supseteq \bar{b}$. However, this contradicts the assumption of this subcase that $\eta_{\hat{m}',r}^z < \mu' = \mu \leq \eta_r^{\hat{m}}$. Hence, we must have $|h_r^z(\bar{b}')| \geq |h_r^z(\bar{b})| + 1$. Then, from (30), we have

$\eta^z_{\hat{m}',r} \geq \theta(|h^z_r(\bar{b}')|) - \theta(|h^z_r(\bar{b}')| - 1) \geq \mu$. Hence, $\bar{V}^z_r(i|\bar{b}') \leq \bar{V}^z_r(i|\bar{b})$.

(2-3) $\eta^z_{\hat{m}',r} \geq \mu'$ and $\eta^z_{\hat{m},r} < \mu$. From (30) and (31), $h^z_r(i \cup \bar{b}') = h^z_r(\bar{b}') \cup \{i\}$ and $h^z_r(\{i\} \cup \bar{b}) = h^z_r(\bar{b}) \setminus \{\hat{m}'\} \cup \{i\}$. Therefore, $\bar{V}^z_r(i|\bar{b}') = \eta^z_{i,r} - \mu'$ and $\bar{V}^z_r(i|\bar{b}) = \eta^z_{i,r} - \eta^z_{\hat{m},r}$. Since $\mu' \geq \mu \geq \eta^z_{\hat{m},r}$, we know that $\bar{V}^z_r(i|\bar{b}') \leq \bar{V}^z_r(i|\bar{b})$.

(2-4) $\eta^z_{\hat{m}',r} < \mu'$ and $\eta^z_{\hat{m},r} < \mu$. From (30) and (31), $h^z_r(\{i\} \cup \bar{b}') = h^z_r(\bar{b}') \setminus \{\hat{m}'\} \cup \{i\}$, and $h^z_r(\{i\} \cup \bar{b}) = h^z_r(\bar{b}) \setminus \{\hat{m}\} \cup \{i\}$. Therefore, $\bar{V}^z_r(i|\bar{b}') = \eta^z_{i,r} - \eta^z_{\hat{m}',r}$ and $\bar{V}^z_r(i|\bar{b}) = \eta^z_{i,r} - \eta^z_{\hat{m},r}$. If $|h^z_r(\bar{b}')| = |h^z_r(\bar{b})|$, then we must have $\eta^z_{\hat{m}',r} \geq \eta^z_{\hat{m},r}$, and hence $\bar{V}^z_r(i|\bar{b}') \leq \bar{V}^z_r(i|\bar{b})$. On the other hand, if $|h^z_r(\bar{b}')| \geq |h^z_r(\bar{b})| + 1$, then from (30) we have $\eta^z_{\hat{m},r} \geq \theta(|h^z_r(\bar{b}')|) - \theta(|h^z_r(\bar{b}')| - 1) \geq \mu > \eta^z_{\hat{m},r}$. Therefore, we can also conclude that $\bar{V}^z_r(i|\bar{b}') \leq \bar{V}^z_r(i|\bar{b})$.

From all four subcases, we can conclude that in case 2, $\bar{V}^z_r(i|\bar{b}) \geq \bar{V}^z_r(i|\bar{b}')$.

We now prove that $\bar{V}^z_r$ satisfies condition (ii) of Definition 4 by contradiction. Assume for the sake of contradiction that Definition 4 (ii) is not satisfied. Then, there must exist a group $\bar{b} \in \bar{B}$, and $i, j, k \in M \setminus \bar{b}$ such that:

$$\bar{V}^z_r(i,j|\bar{b}) + \bar{V}^z_r(k|\bar{b}) > \bar{V}^z_r(i|\bar{b}) + \bar{V}^z_r(j,k|\bar{b}), \quad \Rightarrow \quad \bar{V}^z_r(j|i,\bar{b}) > \bar{V}^z_r(j|k,\bar{b}), \tag{32a}$$

$$\bar{V}^z_r(i,j|\bar{b}) + \bar{V}^z_r(k|\bar{b}) > \bar{V}^z_r(j|\bar{b}) + \bar{V}^z_r(i,k|\bar{b}), \quad \Rightarrow \quad \bar{V}^z_r(i|j,\bar{b}) > \bar{V}^z_r(i|k,\bar{b}). \tag{32b}$$

We consider the following four cases:

Case A: $h^z_r\left(\bar{b} \cup \{i,j\}\right) = h^z_r\left(\bar{b} \cup \{i\}\right) \cup \{j\}$ and $h^z_r\left(\bar{b} \cup \{j,k\}\right) = h^z_r\left(\bar{b} \cup \{k\}\right) \cup \{j\}$. In this case, if $|h^z_r\left(\bar{b} \cup \{i\}\right)| \geq |h^z_r\left(\bar{b} \cup \{k\}\right)|$, then $\bar{V}^z_r(j|i,\bar{b}) \leq \bar{V}^z_r(j|k,\bar{b})$, which contradicts (32a). On the other hand, if $|h^z_r\left(\bar{b} \cup \{i\}\right)| < |h^z_r\left(\bar{b} \cup \{k\}\right)|$, then we must have $h^z_r\left(\bar{b} \cup \{i\}\right) = h^z_r(\bar{b})$ and $h^z_r\left(\bar{b} \cup \{k\}\right) = h^z_r(\bar{b}) \cup \{k\}$. Therefore, $\bar{V}^z_r(i|j,\bar{b}) = 0$, and (32b) cannot hold. We thus obtain the contradiction.

Case B: $|h^z_r\left(\bar{b} \cup \{i,j\}\right)| = |h^z_r\left(\bar{b} \cup \{i\}\right)|$ and $|h^z_r\left(\bar{b} \cup \{j,k\}\right)| = |h^z_r\left(\bar{b} \cup \{k\}\right)|$. We further consider the following four sub-cases:

(B-1). $h^z_r\left(\bar{b} \cup \{i,j\}\right) = h^z_r\left(\bar{b} \cup \{i\}\right)$ and $h^z_r\left(\bar{b} \cup \{j,k\}\right) = h^z_r\left(\bar{b} \cup \{k\}\right)$. In this case, $\bar{V}^z_r(j|i,\bar{b}) = \bar{V}^z_r(j|k,\bar{b}) = 0$. Hence, we arrive at a contradiction against (32a).

(B-2). $h^z_r\left(\bar{b} \cup \{i,j\}\right) \neq h^z_r\left(\bar{b} \cup \{i\}\right)$ and $h^z_r\left(\bar{b} \cup \{j,k\}\right) = h^z_r\left(\bar{b} \cup \{k\}\right)$. In this case, when j is added to the set $\bar{b} \cup \{i\}$, j replaces an agent, denoted as $\hat{m} \in \bar{b} \cup \{i\}$. Since $\hat{m}$ is replaced, we must have $\eta^z_{\hat{m},r} \leq \eta^z_{m,r}$ for any $m \in h^z_r(\bar{b} \cup \{j\})$. If $\hat{m} = i$, then $h^z_r(\bar{b} \cup \{i,j\}) = h^z_r(\bar{b} \cup \{j\})$. Hence, $\bar{V}^z_r(i|j,\bar{b}) = 0$, and we arrive at a contradiction with (32b). On the other hand, if $\hat{m} \neq i$, then $\hat{m}$ is an agent in group $\bar{b}$. This implies that $\hat{m} \in \bar{b}$ should be replaced by j when j is added to the set $\{k\} \cup \bar{b}$, which contradicts the assumption of this case that $h^z_r\left(\bar{b} \cup \{j,k\}\right) = h^z_r\left(\bar{b} \cup \{k\}\right)$.

(B-3). $h^z_r\left(\bar{b} \cup \{i,j\}\right) = h^z_r\left(\bar{b} \cup \{i\}\right)$ and $h^z_r\left(\bar{b} \cup \{j,k\}\right) \neq h^z_r\left(\bar{b} \cup \{k\}\right)$. Analogous to case *B-2*, we know that $h^z_r\left(\bar{b} \cup \{j,k\}\right) = h^z_r\left(\bar{b} \cup \{j\}\right)$ and $\eta^z_{j,r} \geq \eta^z_{k,r}$. Moreover, since

$h_r^z\left(\bar{b}\cup\{i,j\}\right)=h_r^z\left(\bar{b}\cup\{i\}\right)$, we must have $\eta_{j,r}^z\le\eta_{i,r}^z$. Therefore, $\bar{V}_r^z(\bar{b}\cup\{i,j\})=\bar{V}_r^z(\bar{b}\cup\{i\})$, and $\bar{V}_r^z(i|j,\bar{b})=\bar{V}_r^z(\bar{b}\cup\{i\})-\bar{V}_r^z(\bar{b}\cup\{j\})$. Since $\eta_{j,r}^z\le\eta_{i,r}^z$ and $\eta_{j,r}^z\ge\eta_{k,r}^z$, we know that $\bar{V}_r^z(i|k,\bar{b})=\bar{V}_r^z(\bar{b}\cup\{i\})-\bar{V}_r^z(\bar{b}\cup\{k\})\ge\bar{V}_r^z(\bar{b}\cup\{i\})-\bar{V}_r^z(\bar{b}\cup\{j\})=\bar{V}_r^z(i|j,\bar{b})$, which contradicts (32b).

(B-4). $h_r^z\left(\bar{b}\cup\{i,j\}\right)\neq h_r^z\left(\bar{b}\cup\{i\}\right)$ and $h_r^z\left(\bar{b}\cup\{j,k\}\right)\neq h_r^z\left(\bar{b}\cup\{k\}\right)$. In this case, if $h_r^z\left(\bar{b}\cup\{i,j\}\right)=h_r^z\left(\bar{b}\cup\{j\}\right)$, then $\bar{V}_r^z(i|j,\bar{b})=\bar{V}_r^z(i,j,\bar{b})-\bar{V}_r^z(j,\bar{b})=\bar{V}_r^z(j,\bar{b})-\bar{V}_r^z(j,\bar{b})=0$, which contradicts (32b). On the other hand, if $h_r^z\left(\bar{b}\cup\{i,j\}\right)\neq h_r^z\left(\bar{b}\cup\{j\}\right)$, then one agent $\hat{m}\in\bar{b}$ must be replaced by j when j is added into the set $\bar{b}\cup\{i\}$, i.e. $h_r^z\left(\bar{b}\cup\{i,j\}\right)=h_r^z\left(\bar{b}\setminus\{\hat{m}\}\cup\{i,j\}\right)$. Hence, $\eta_{\hat{m},r}^z\le\eta_{i,r}^z$ and $\eta_{\hat{m},r}^z\le\eta_{j,r}^z$. If $\eta_{\hat{m},r}^z\le\eta_{k,r}^z$, then under the assumption that $|h_r^z\left(\bar{b}\cup\{j,k\}\right)|=|h_r^z\left(\bar{b}\cup\{k\}\right)|$ and $h_r^z\left(\bar{b}\cup\{j,k\}\right)\neq h_r^z\left(\bar{b}\cup\{k\}\right)$, we must have $h_r^z\left(\bar{b}\cup\{j,k\}\right)=h_r^z\left(\bar{b}\setminus\{\hat{m}\}\cup\{j,k\}\right)$. Then, we can check that $\bar{V}_r^z(j|i,b)=\bar{V}_r^z(j|k,b)$, which contradicts (32a).

On the other hand, if $\eta_{\hat{m},r}^z>\eta_{k,r}^z$, then $h_r^z\left(\bar{b}\cup\{j,k\}\right)=h_r^z\left(\bar{b}\cup\{j\}\right)$. In this case, $\bar{V}_r^z(i|j,\bar{b})$ is the change of trip value by replacing $\hat{m}$ with i, and $\bar{V}_r^z(i|k,\bar{b})$ is the change of trip value by replacing k with i. Since $\eta_{k,r}^z<\eta_{\hat{m},r}^z$, we must have $\bar{V}_r^z(i|j,\bar{b})<\bar{V}_r^z(i|k,\bar{b})$, which contradicts (32b).

Case C: $h_r^z\left(\bar{b}\cup\{i,j\}\right)=h_r^z\left(\bar{b}\cup\{i\}\right)\cup\{j\}$ and $|h_r^z\left(\bar{b}\cup\{j,k\}\right)|=|h_r^z\left(\bar{b}\cup\{k\}\right)|$. We further consider the following sub-cases:

(C-1). $h_r^z\left(\bar{b}\cup\{j,k\}\right)=h_r^z\left(\bar{b}\cup\{k\}\right)$. In this case, $\eta_{j,r}^z\le\eta_{m,r}^z$ for all $m\in h_r^z(\bar{b}\cup\{k\})$, and $\eta_{j,r}^z<\theta(|h_r^z(\bar{b}\cup\{k\})+1|)-\theta(|h_r^z(\bar{b}\cup\{k\})|)$. Since $h_r^z\left(\bar{b}\cup\{i,j\}\right)=h_r^z\left(\bar{b}\cup\{i\}\right)\cup\{j\}$, we know that $\eta_{j,r}^z\ge\theta(|h_r^z(\bar{b}\cup\{i\})+1|)-\theta(|h_r^z(\bar{b}\cup\{i\})|)$. Since disutility of trip sharing is non-decreasing in agent group size, for $\eta_{j,r}^z$ to satisfy both inequalities, we must have $|h_r^z(\bar{b}\cup\{i\})|<|h_r^z(\bar{b}\cup\{k\})|$. Then, we must have $h_r^z(\bar{b}\cup\{i\})=h_r^z(\bar{b})$ and $h_r^z(\bar{b}\cup\{k\})=h_r^z(\bar{b})\cup\{k\}$. Therefore, $\bar{V}_r^z(i,j,\bar{b})=\bar{V}_r^z(j,\bar{b})$ and $\bar{V}_r^z(i,k,\bar{b})=\bar{V}_r^z(k,\bar{b})$. Hence, $\bar{V}_r^z(i|j,\bar{b})=\bar{V}_r^z(i|k,\bar{b})=0$, which contradicts (32b).

(C-2). $h_r^z\left(\bar{b}\cup\{j,k\}\right)\neq h_r^z\left(\bar{b}\cup\{k\}\right)$. Since $|h_r^z\left(\bar{b}\cup\{j,k\}\right)|=|h_r^z\left(\bar{b}\cup\{k\}\right)|$, j replaces an agent $\hat{m}$ in $\bar{b}\cup\{k\}$, and $\eta_{\hat{m},r}^z\le\eta_{m,r}^z$ for all $m\in\bar{b}\cup k$. If $\hat{m}=k$, then $h_r^z\left(\bar{b}\cup\{j,k\}\right)=h_r^z\left(\bar{b}\cup\{j\}\right)$. Therefore, $\bar{V}_r^z(j|i,\bar{b})=\eta_{j,r}^z-\left(\theta(|h_r^z(\bar{b}\cup\{i\})|+1)-\theta(|h_r^z(\bar{b}\cup\{i\})|)\right)$ and $\bar{V}_r^z(j|k,\bar{b})=\eta_{j,r}^z-\eta_r^k$. If $\eta_{k,r}^z\le\theta(|h_r^z(\bar{b}\cup\{i\})|+1)-\theta(|h_r^z(\bar{b}\cup\{i\})|)$, then (32a) is contradicted. Thus, $\eta_{k,r}^z>\theta(|h_r^z(\bar{b}\cup\{i\})|+1)-\theta(|h_r^z(\bar{b}\cup\{i\})|)$. Since k is replaced by j when j is added to $\bar{b}\cup\{k\}$, we must have $\eta_{k,r}^z<\theta(|h_r^z(\bar{b}\cup\{j\})|+1)-\theta(|h_r^z(\bar{b}\cup\{j\})|)$. For $\eta_{k,r}^z$ to satisfy both inequalities, we must have $|h_r^z(\bar{b}\cup\{j\})|>|h_r^z(\bar{b}\cup\{i\})|$. Hence, $h_r^z(\bar{b}\cup\{j\})=h_r^z(\bar{b})\cup\{j\}$ and $h_r^z(\bar{b}\cup\{i\})=h_r^z(\bar{b})$. Then, $\bar{V}_r^z(i|j,\bar{b})=\bar{V}_r^z(\bar{b}\cup\{i,j\})-\bar{V}_r^z(\bar{b}\cup\{j\})=0$, which contradicts (32b).

On the other hand, if $\hat{m}\in\bar{b}$, then we know from (31) that $\eta_{\hat{m},r}^z<\theta(|h_r^z\left(\bar{b}\cup\{k\}\right)|+1)-\theta(|h_r^z\left(\bar{b}\cup\{k\}\right)|)$. Additionally, since $h_r^z\left(\bar{b}\cup\{i,j\}\right)=h_r^z\left(\bar{b}\cup\{i\}\right)\cup\{j\}$, we know from

(30) that $\eta^z_{\hat{m},r} \geq \theta(|h^z_r\left(\bar{b} \cup \{i\}\right)| + 1) - \theta(|h^z_r\left(\bar{b} \cup \{i\}\right)|)$. If $\eta^z_{\hat{m},r}$ satisfies both inequalities, then we must have $|h^z_r\left(\bar{b} \cup \{i\}\right)| < |h^z_r\left(\bar{b} \cup \{k\}\right)|$. Therefore, $h^z_r\left(\bar{b} \cup \{i\}\right) = h^z_r(\bar{b})$. Then, $\bar{V}^z_r(i|j, \bar{b}) = 0$, which contradicts (32b).

Case D: $|h^z_r\left(\bar{b} \cup \{i, j\}\right)| = |h^z_r\left(\bar{b} \cup \{i\}\right)|$ and $h^z_r\left(\bar{b} \cup \{j, k\}\right) = h^z_r\left(\bar{b} \cup \{k\}\right) \cup \{j\}$. We further consider the following sub-cases:

(D-1). $h^z_r\left(\bar{b} \cup \{i, j\}\right) = h^z_r\left(\bar{b} \cup \{i\}\right)$. In this case, analogous to *(C-1)*, we know that $|h^z_r(\bar{b} \cup \{k\})| < |h^z_r(\bar{b} \cup \{i\})|$. Therefore, $h^z_r(\bar{b} \cup \{k\}) = h^z_r(\bar{b})$ and $h^z_r(\bar{b} \cup \{i\}) = h^z_r(\bar{b}) \cup \{i\}$. Therefore, $\eta^z_{k,r} < \eta^z_{i,r}$. Additionally, since $h^z_r\left(\bar{b} \cup \{i, j\}\right) = h^z_r\left(\bar{b} \cup \{i\}\right)$, $\eta^z_{j,r} < \eta^z_{i,r}$. Then, $\bar{V}^z_r(i|j, \bar{b}) = \bar{V}^z_r(i, \bar{b}) - \bar{V}^z_r(j, \bar{b})$ and $\bar{V}^z_r(i|k, \bar{b}) = \bar{V}^z_r(i, \bar{b}) - \bar{V}^z_r(\bar{b})$. Since $\bar{V}^z_r$ is monotonic, $\bar{V}^z_r(j, \bar{b}) \geq \bar{V}^z_r(\bar{b})$ so that $\bar{V}^z_r(i|j, \bar{b}) \leq \bar{V}^z_r(i|k, \bar{b})$, which contradicts (32b).

(D-2). $h^z_r\left(\bar{b} \cup \{i, j\}\right) \neq h^z_r\left(\bar{b} \cup \{i\}\right)$. Since $|h^z_r\left(\bar{b} \cup \{i, j\}\right)| = |h^z_r\left(\bar{b} \cup \{i\}\right)|$, j replaces the agent $\hat{m} \in \bar{b} \cup \{i\}$ such that $\eta^z_{\hat{m},r} \leq \eta^z_{m,r}$ for all $m \in h^z_r(\bar{b} \cup \{i\})$. If $\hat{m} = i$, then analogous to case *C-2*, we know that if (32b) is satisfied, then $|h^z_r(\bar{b} \cup \{j\})| < |h^z_r(\bar{b} \cup \{k\})|$. Hence, $h^z_r(\bar{b} \cup \{j\}) = h^z_r(\bar{b})$ and $V(j|i, \bar{b}) = 0$, which contradicts (32a).

On the other hand, if $\hat{m} \in \bar{b}$, then again analogous to case *C-2*, we know that $|h^z_r\left(\bar{b} \cup \{k\}\right)| < |h^z_r\left(\bar{b} \cup \{i\}\right)|$. Therefore, $h^z_r\left(\bar{b} \cup \{k\}\right) = h^z_r(\bar{b})$, and $h^z_r\left(\bar{b} \cup \{i\}\right) = h^z_r(\bar{b}) \cup \{i\}$. Then, $\bar{V}^z_r(j|i, \bar{b}) = \bar{V}^z_r(\bar{b} \setminus \{\hat{m}\} \cup \{i, j\}) - \bar{V}^z_r(i, \bar{b})$, and $\bar{V}^z_r(j|k, \bar{b}) = \bar{V}^z_r(\bar{b} \cup \{j\}) - \bar{V}^z_r(\bar{b})$. Since $\hat{m} \neq i$, $\bar{V}^z_r(i|j, \bar{b}) = \bar{V}^z_r(\bar{b} \setminus \{\hat{m}\} \cup \{i, j\}) - \bar{V}^z_r(j, \bar{b}) = \eta^z_{i,r} - \eta^z_{\hat{m},r}$. Additionally, since $h^z_r(i, \bar{b}) = h^z_r(\bar{b}) \cup \{i\}$, $\bar{V}^z_r(i|k, \bar{b}) = \bar{V}^z_r(i, \bar{b}) - \bar{V}^z_r(\bar{b}) = \eta^z_{i,r} - (\theta(|h^z_r(\bar{b})| + 1) - \theta(|h^z_r(\bar{b})|))$. Since $h^z_r(\bar{b} \cup \{i\}) = h^z_r(\bar{b}) \cup \{i\}$ and $\hat{m} \in \bar{b}$, we know from (30) that $\eta^z_{\hat{m},r} \geq \theta(|h^z_r(\bar{b})| + 1) - \theta(|h^z_r(\bar{b})|)$. Therefore, $\bar{V}^z_r(i|j, \bar{b}) \leq \bar{V}^z_r(i|k, \bar{b})$, which contradicts (32b).

From all above four cases, we can conclude that condition *(ii)* of Definition 4 is satisfied. We can thus conclude that $\bar{V}^z_r$ satisfies gross substitutes condition. $\square$

Proof of Lemma 3. For any route $r \in R^* = \{R | w^*_r > 0\}$ and any $z = 1, \ldots, T - d_r$, we denote L^z_r as the set of slots that correspond to departing at time z and using route r. From the temporally repeated flow vector k^*, we know that $|L^z_r| = k^{z*}_r$. We denote $L = \cup_{r \in R^*} \cup_{z=1}^{T-d_r} L^z_r$. Given any optimal integer solution x^* of $(\overline{\text{LP}}k^*)$, we denote $\{\bar{b}_l\}_{l \in L^z_r}$ as the set of augmented groups in $\bar{B}$ such that $\bar{x}^{z*}_r(\bar{b}) = 1$. In particular, if the number of agent groups that take route r with departure time z is less than k^{z*}_r, then $\bar{b}_l = \emptyset$ for some of $l \in L^z_r$.

We first show that $\bar{x}^*$ is an integer optimal solution of $(\overline{\text{LP}}k^*)$ if and only if $((\bar{b}_l)_{l \in L}, u^*)$ is a Walrasian equilibrium of the equivalent economy with good set M and buyer set L. Here, u^* is the optimal dual variable in the dual program of $(\overline{\text{LP}}k^*)$ associated with constraint $(\overline{\text{LP}}k^*.\text{a})$. If $((\bar{b}_l)_{l \in L}, u^*)$ is a Walrasian equilibrium of the equivalent economy, then the associated $\bar{x}^*$ must be a feasible solution of $(\overline{\text{LP}}k^*)$. We define $\lambda^{z*}_r = \bar{V}_l(\bar{b}_l) - \sum_{m \in \bar{b}_l} u^*_m = \max_{\bar{b} \in \bar{B}}\{\bar{V}_l(\bar{b}) - \sum_{m \in \bar{b}} u^*_m\}$ for $l \in L^z_r$, where the second equality follows from the definition of Walrasian equilibrium. We can check that (u^*, τ^*) satisfies the dual constraints of

($\overline{\text{LP}}k^*$), and (x^*, u^*, τ^*) satisfies the complementary slackness conditions associated with all the primal and dual constraints. Thus, x^* is an optimal integer solution of ($\overline{\text{LP}}k^*$). On the other hand, we can analogously argue that if x^* is an optimal integer solution of ($\overline{\text{LP}}k^*$), then the associated $(\bar{b}_l)_{l\in L}$ and the dual optimal solution u^* is a Walrasian equilibrium in the equivalent economy.

From Lemma 4, we know that when the augmented value function $\bar{V}_l = \bar{V}_r^z$ satisfies the monotonicity and gross substitutes conditions, Walrasian equilibrium exists in the equivalent economy. As a result, we know that the associated $\bar{x}^*$ is an optimal integer solution in ($\overline{\text{LP}}k^*$).

Finally, in (18), we select one representative agent group $h_r^z(\bar{b})$ for each $\bar{b}$ that is assigned to (r, z) as the true feasible agent group that takes route r at time z. Such x^* achieves the same social welfare as that in $\bar{x}^*$, and thus is an optimal solution of (LPk^*). □

E Proof of Statements in Section 3.2

We define $U^* \triangleq \{u | \exists \tau \text{ such that } (u, \tau) \text{ is optimal solution of (D)}\}$ as the equilibrium utility set.

Lemma 9. *If the network is series-parallel, a utility vector $u^* \in U^*$ if and only if there exists a vector λ^* such that (u^*, λ^*) is an optimal solution of* (Dk^*).

We can check that for any optimal solution (u^*, τ^*) of (D), the vector (u^*, λ^*) – where $\lambda_r^{z*} = \sum_{e\in r} \tau_e^{z+\theta_{r,e}*}$ for each $r \in R$ – must also be optimal in (Dk^*). That is, any $u^* \in U^*$ is also an optimal utility vector in (Dk^*). On the other hand, we show that any optimal utility vector of (Dk^*) is also an equilibrium utility vector in U^* in that we can find an edge price vector τ^* such that (u^*, τ^*) is an optimal solution of (D). We prove this argument by mathematical induction using the series-parallel network condition: If such an equilibrium edge price vector exists on two series parallel networks, then the combined edge price vector is also an equilibrium edge price vector when the two networks are connected in series or in parallel.

Lemma 9 enables us to characterize the riders' equilibrium utility set U^* using the dual program (Dk^*) associated with the optimal trip organization problem under the capacity constraint with k^*. The following lemma further shows that U^* is a lattice with $u^\dagger$ being the maximum element.

Lemma 10. *If the network is series-parallel, and riders have homogeneous carpool disutilities, then the set U^* is a complete lattice with $u^\dagger \in U^*$, and $u_m^\dagger \geq u_m^*$ for any $u^* \in U^*$.*

The proof of Lemma 10, utilizes the equivalence between the trip organization problem with the augmented trip value function $\bar{V}$ and the economy constructed in Sec. 3. In particular, we can show that U^* characterized by (Dk^*) is identical to set of good prices in Walrasian equilibrium of the economy. From Lemma 2, we know that $\bar{V}$ satisfy monotonicity and gross substitutes conditions when riders have homogeneous carpool disutilities. Building on the theory of Walrasian equilibrium (Gul and Stacchetti [1999], also included in Lemma 5 in Appendix B), we can show that U^* is a lattice, and $u^\dagger$ is the maximum element in U^*.

Combining Lemmas 9 and 10, we know that $(x^*, p^\dagger, \tau^\dagger)$ is a market equilibrium. We conclude Theorem 2 by noticing that $(x^*, p^\dagger)$, where $p^\dagger$ is the VCG price as in (19), implements the same outcome as a VCG mechanism, and thus $(x^*, p^\dagger, \tau^\dagger)$ must be strategyproof.

Proof of Lemma 9. We first show that for any optimal utility vector $u^* \in U^*$, there exists a vector λ^* such that (u^*, λ^*) is an optimal solution of (Dk^*). Since $u^* \in U^*$, there must exist an edge price vector τ^* such that (u^*, τ^*) is an optimal solution of (D). Consider $\lambda^* = (\lambda_r^{z*})_{r \in R^*, z=1,\ldots,T}$ as follows:

$$\lambda_r^{z*} = \sum_{e \in r} \tau_e^{z+d_{r,e}*}, \quad \forall r \in R, \quad \forall z = 1, \ldots T. \tag{33}$$

Since (u^*, τ^*) is feasible in (D), we can check that (u^*, λ^*) is also a feasible solution of (Dk^*). Moreover, since (x^*, u^*, τ^*) satisfies complementary slackness conditions with respect to (LP) and (D), (x^*, u^*, λ^*) also satisfies complementary slackness conditions with respect to (LPk^*) and (Dk^*). Therefore, (u^*, λ^*) is an optimal solution of (Dk^*).

We next show that for any optimal solution (u^*, λ^*) of (Dk^*), we can find an edge price vector τ^* such that (u^*, τ^*) is an optimal solution of (D) (i.e. $u^* \in U^*$). We prove this argument by mathematical induction. To begin with, if the network only has a single edge $E = \{e\}$, then for any optimal solution (u^*, λ^*), we can check that (u^*, τ^*) where $\tau_e^{z*} = \lambda_e^{z*}$ is an optimal solution of (D). We now prove that if this argument holds on two series-parallel networks G^1 and G^2, then it also holds on the network constructed by connecting G^1 and G^2 in parallel or in series. We prove the case of parallel connection and series connection separately as follows:

(Case 1). The network G is constructed by connecting G^1 and G^2 in parallel. In each network G^i $(i = 1, 2)$, we define E^i as the set of edges, R^i as the set of routes. We also define k^{i*} as the optimal temporally repeated flow vector computed from Alg. 1 in G^i. Since G^1 and G^2 are connected in parallel, we have $E^1 \cup E^2 = E$, $R^1 \cup R^2 = R$, and $k^* = (k^{1*}, k^{2*})$.

For each $i = 1, 2$, we consider the sub-problem, where riders organize trips on the sub-network G^i. For any (x^*, u^*, λ^*) on the original network G, we define the trip vector $x^{i*} = \left(x_{r^i}^{z^i*}(b)\right)_{r^i \in R^i, b \in B, z^i=1,\ldots T}$ and the route price vector $\lambda^{i*} = \left(\lambda_{r^i}^{z^i*}\right)_{r^i \in R^i, z^i=1,\ldots T}$ for the subnetwork G^i. We can check that the vector x^{i*} is a feasible solution of (LPk^*) for the subproblem, where k^* in the original problem (LPk^*) is replaced by k^{i*}. Additionally, the vector (u^*, λ^{i*}) is a feasible solution of (Dk^*). Since the original optimal solutions x^* and (u^*, λ^*) satisfy the complementary slackness conditions of constraints (LPk^*.a)-(LPk^*.b) and (Dk^*.a) for all $m \in M$ and all $r \in R = R^1 \cup R^2$, we know that x^{i*} and (u^*, λ^{i*}) must also satisfy the complementary slackness conditions of these constraints in each subproblem. Therefore, x^{i*} is an optimal integer solution of (LPk^*) and (u^*, λ^{i*}) is an optimal solution of (Dk^*) in the subproblem.

From our assumption of mathematical induction, there exists an edge price vector $\tau^{i*} = (\tau_e^{t,i*})_{e \in E^i, t=1,\ldots,T}$ such that (u^*, τ^{i*}) is an optimal solution of (D) in each subproblem i with subnetwork G^i. Thus, (u^*, τ^{i*}) satisfies the feasibility constraints in (D) of each subproblem i, and x^{i*} and (u^*, τ^{i*}) satisfy the complementary slackness conditions with respect to constraints (LP.a) for each $m \in M$, (LP.b) for each $e \in E^i$, (D.a) for each $r^i \in R^i$. Consider the edge price vector $\tau^* = (\tau^{1*}, \tau^{2*})$. Since $R = R^1 \cup R^2$ and $E = E^1 \cup E^2$, (u^*, τ^*) must be feasible in (D) on the original network, and x^*, (u^*, τ^*) must satisfy the complementary slackness conditions with respect to constraints (LP.a) – (LP.b), and (D.a). Therefore, we can conclude that for any optimal solution (u^*, λ^*) of (Dk^*), there exists an edge price vector τ^* such that (u^*, τ^*) is an optimal solution of (D) in network G.

(Case 2). The network G is constructed by connecting G^1 and G^2 in series, where G^1 is the subnetwork connected to the origin. Same as that in case 1, we define E^i as the set of edges in the subnetwork G^i $(i = 1, 2)$, and R^i as the set of routes. Since G^1 and G^2 are connected in series, we have $E = E^1 \cup E^2$, and $R = R^1 \times R^2$.

We define a sub-trip (z^i, b, r^i) as the trip in the sub-network G^i where rider group b takes route $r^i \in R^i$, and departs from the origin of network G^i at time z^i. Analogous to the value of trip defined in (3), we define the value of each sub-trip (z^i, b, r^i) as follows:

$$V_{r^1}^{z^1}(b) = \sum_{m \in b} \left(\alpha_m^1 - \beta_m d_{r^1}\right) - |b| \left(\pi(|b|) + \gamma(|b|) d_{r^1} + \sigma + \delta d_{r^1}\right), \quad \forall b \in B,\ \forall r^1 \in R^1,\ \forall z^1 = 1, \ldots, T, \tag{34}$$

$$V_{r^2}^{z^2}(b) = \sum_{m \in b} \left(\alpha_m^2 - \beta_m d_{r^2} - \ell_m((z^2 + d_{r^2} - \theta_m)_+)\right) - |b| \left(\pi(|b|) + \gamma(|b|) d_{r^2} + \sigma + \delta d_{r^2}\right), \quad \forall b \in B,\ \forall r^2 \in R^2,\ \forall z^2 = 1, \ldots, T. \tag{35}$$

where α_m^i is the value for rider m to travel from the origin to the destination of the subnetwork G^i. The value of α_m^i can be any number in $[0, \alpha_m]$ as long as $\alpha_m^1 + \alpha_m^2 = \alpha_m$. We note that the delay cost of the original trip is included entirely in the second sub-trip in G^2. We can check that $V_{r^1}^{z^1}(b) + V_{r^2}^{z^1+d_{r^1}}(b) = V_{r^1r^2}^{z^1}(b)$ is the value of the entire trip (z^1, b, r^1r^2) of the original network.

We denote the trip organization vector on G^i as $x^i = \left(x_{r^i}^{z^i}(b)\right)_{r^i\in R^i, b\in B, z^i=1,\ldots,T}$, where $x_{r^i}^{z^i}(b) = 1$ if the sub-trip (z^i, b, r^i) is organized in G^i, and 0 otherwise. The optimal trip organization problem (LP) can be equivalently presented by (x^1, x^2) as follows:

$$
\begin{aligned}
\max_{x^1,x^2} \quad & S(x^1,x^2) = \sum_{z^1=1}^{T}\sum_{b\in B}\sum_{r^1\in R^1} V_{r^1}^{z^1}(b)x_{r^1}^{z^1}(b) + \sum_{z^2=1}^{T}\sum_{b\in B}\sum_{r^2\in R^2} V_{r^2}^{z^2}(b)x_{r^2}^{z^2}(b) \\
s.t. \quad & \sum_{r^i\in R^i}\sum_{z^i=1}^{T}\sum_{b\ni m} x_{r^i}^{z^i}(b) \le 1, \quad \forall m \in M, \quad \forall i = 1,2, && \text{(36a)} \\
& \sum_{r^i\ni e}\sum_{b\in B} x_{r^i}^{t-d_{r^i,e}}(b) \le q_e, \quad \forall e \in E^i, \quad \forall i = 1,2, \quad \forall t = 1,\ldots T, && \text{(36b)} \\
& \sum_{r^1\in R^1} x^{t-d_{r^1}}(b) = \sum_{r^2\in R^2} x_{r^2}^{t}(b), \quad \forall b \in B, \quad \forall t = 1,\ldots,T, && \text{(36c)} \\
& x_{r^i}^{z^i}(b) \ge 0, \quad \forall b \in B, \quad \forall r^i \in R^i, \quad \forall i = 1,2, \quad \forall z^i = 1,\ldots,T, && \text{(36d)}
\end{aligned}
$$

where (36a) and (36b) are the constraints of x^i in the trip organization sub-problem on G^i. The constraint (36c) ensures that any rider group that arrive at the destination of G^1 at time t must depart from the origin of G^2 at time t to complete a trip in the original network G.

We denote k^* as the optimal capacity vector computed from Alg. 1. Since Alg. 1 allocates capacity on routes in increasing order of their travel time, and the total travel time of each route is $d_{r^1r^2} = d_{r^1} + d_{r^2}$, we know that $k_{r^1}^{z^1*} = \sum_{r^2\in R^2} k_{r^1r^2}^{z^1*}$ for all $r^1 \in R^1$ and $k_{r^2}^{z^2*} = \sum_{r^1\in R^1} k_{r^1r^2}^{z^2-d_{r^1}*}$ for all $r^2 \in R^2$. Analogous to the proof of Lemma 1, any optimal integer solution of the following linear program is an optimal solution of (36):

$$
\begin{aligned}
\max_{x^1,x^2} \quad & S(x^1,x^2) = \sum_{z^1=1}^{T}\sum_{b\in B}\sum_{r^1\in R^1} V_{r^1}^{z^1}(b)x_{r^1}^{z^1}(b) + \sum_{z^2=1}^{T}\sum_{b\in B}\sum_{r^2\in R^2} V_{r^2}^{z^2}(b)x_{r^2}^{z^2}(b) \\
s.t. \quad & \sum_{r^i\in R^i}\sum_{z^i=1}^{T}\sum_{b\ni m} x_{r^i}^{z^i}(b) \le 1, \quad \forall m \in M, \quad \forall i = 1,2, && \text{(37a)} \\
& \sum_{b\in B} x_{r^i}^{z^i}(b) \le k_{r^i}^{z^i*}, \quad \forall r^i \in R^i, \quad \forall i = 1,2, \quad \forall t = 1,\ldots T, && \text{(37b)} \\
& \sum_{r^1\in R^1} x_{r^1}^{t-d_{r^1}}(b) = \sum_{r^2\in R^2} x_{r^2}^{t}(b), \quad \forall b \in B, \quad \forall t = 1,\ldots,T, && \text{(37c)} \\
& x_{r^i}^{z^i}(b) \ge 0, \quad \forall b \in B, \quad \forall r^i \in R^i, \quad \forall i = 1,2, \quad \forall z^i = 1,\ldots,T, && \text{(37d)}
\end{aligned}
$$

We note that a trip (z, b, r^1r^2) is organized if and only if both $x^z_{r^1}(b) = 1$ and $x^{z+d_{r^1}}_{r^2}(b) = 1$. Thus, any (x^1, x^2) is feasible in (36) (resp. (37)) if and only if there exists a feasible x in (LP) (resp. (LPk^*)) such that $x^{z^1}_{r^1}(b) = \sum_{r^2\in R^2} x^{z^1}_{r^1r^2}(b)$ and $x^{z^2}_{r^2}(b) = \sum_{r^1\in R^1} x^{z^2-d_{r^1}}_{r^1r^2}(b)$ for all $b \in B$, all $r^1 \in R^1$, $r^2 \in R^2$ and all $z^1, z^2 = 1, \ldots, T$. Moreover, the value of the objective function $S(x^1, x^2)$ equals to $S(x)$ with the corresponding x:

$$\begin{aligned}
S(x) &= \sum_{z=1}^{T}\sum_{b\in B}\sum_{r\in R} V^z_r(b)x^z_r(b) = \sum_{z=1}^{T}\sum_{b\in B}\sum_{r^1\in R^1}\sum_{r^2\in R^2}\left(V^z_{r^1}(b) + V^{z+d_{r^1}}_{r^2}(b)\right)x^z_{r^1r^2}(b)\\
&= \sum_{z^1=1}^{T}\sum_{b\in B}\sum_{r^1\in R^1} V^{z^1}_{r^1}(b)\left(\sum_{r^2\in R^2} x^{z^1}_{r^1r^2}(b)\right) + \sum_{z^2=1}^{T}\sum_{b\in B}\sum_{r^2\in R^2} V^{z^2}_{r^2}(b)\left(\sum_{r^1\in R^1} x^{z^2-d_{r^1}}_{r^1r^2}(b)\right)\\
&= S(x^1, x^2).
\end{aligned}$$

Therefore, given any optimal solution x^* of (LPk^*), the corresponding (x^{1*}, x^{2*}) is an optimal integer solution of (37). Additionally, (x^{1*}, x^{2*}) is also an optimal solution of (36). Hence, the optimal values of (LP), (36), (LPk^*) and (37) are the same.

We introduce the dual variables $u^i = (u^i_m)_{m\in M, i=1,2}$ for constraints (36a), $\tau^i = (\tau^{t,i}_e)_{e\in E^i, t=1,\ldots,T}$ for (36b) of each $i = 1, 2$, and $\chi = (\chi^t(b))_{b\in B, t=1,\ldots,T}$ for (36c). Then, the dual program of (36) can be written as follows:

$$\min_{u,\tau} \quad U(u,\tau) = \sum_{i=1,2}\sum_{m\in M} u^i_m + \sum_{t=1}^{T}\sum_{i=1,2}\sum_{e\in E^i} q_e\tau^{t,i}_e$$

$$s.t. \quad \sum_{m\in b} u^1_m + \sum_{e\in r^1} \tau_e^{z^1+d_{r^1,e}} + \chi^{z^1+d_{r^1}}(b) \geq V^{z^1}_{r^1}(b), \quad \forall z^1 = 1, \ldots, T, \quad \forall b \in B, \quad \forall r^1 \in R^1, \tag{38a}$$

$$\sum_{m\in b} u^2_m + \sum_{e\in r^2} \tau_e^{z^2+d_{r^2,e}} - \chi^{z^2}(b) \geq V^{z^2}_{r^2}(b), \quad \forall z^2 = 1, \ldots, T, \quad \forall b \in B, \quad \forall r^2 \in R^2, \tag{38b}$$

$$u_m \geq 0, \quad \tau^t_e \geq 0, \quad \forall m \in M, \quad \forall e \in E, \quad \forall t = 1, \ldots, T. \tag{38c}$$

Similarly, we obtain the dual program of (37) with the same dual variables except for the route price vector $\lambda^i = \left(\lambda^{t,i}_{r^i}\right)_{r^i\in R^{i*}, t=1,\ldots,T, i=1,2}$ for (37b):

$$\min_{u^1,u^2,\lambda^1,\lambda^2,\chi} \quad U = \sum_{i=1,2}\sum_{m\in M} u^i_m + \sum_{i=1,2}\sum_{z^i=1}^{T}\sum_{r^i\in R^i} k^{z^i*}_{r^i}\lambda^{z^i}_{r^i},$$

$$s.t. \quad \sum_{m\in b} u^1_m + \lambda^{z^1}_{r^1} + \chi^{z^1+d_{r^1}}(b) \geq V^{z^1}_{r^1}(b), \quad \forall z^1 = 1,\dots,T, \quad \forall b\in B, \quad \forall r^1 \in R^1, \tag{39a}$$

$$\sum_{m\in b} u^2_m + \lambda^{z^2}_{r^2} - \chi^{z^2}(b) \geq V^{z^2}_{r^2}(b), \quad \forall z^2 = 1,\dots,T, \quad \forall b\in B, \quad \forall r^2 \in R^2, \tag{39b}$$

$$u^i_m,\ \lambda^{z^i}_{r^i} \geq 0, \quad \forall m \in M, \quad \forall r^i \in R^i, \quad \forall z^i = 1,\dots,T, \quad i = 1,2. \tag{39c}$$

From strong duality, we know that the optimal value of (39) (resp. (Dk^*)) is the same as that of (37) (resp. (LPk^*)). Since the optimal values of (LPk^*) and (37) are identical, we know that the optimal values of (39) must be equal to that of (Dk^*). Additionally, we can check that for any feasible solution $(u^1, u^2, \lambda^1, \lambda^2, \chi)$ of (39) must correspond to a feasible solution (u, λ) of (Dk^*) such that $u_m = u^1_m + u^2_m$ and $\lambda^z_{r^1r^2} = \lambda^z_{r^1} + \lambda^{z+d_{r^1}}_{r^2}$. Then, for each (u^*, λ^*), we consider the optimal solution $(u^{1*}, u^{2*}, \lambda^{1*}, \lambda^{2*}, \chi^*)$ of (39), and define $\tilde{V}^1_{r^1}(b) = V^{z^1}_{r^1}(b) - \chi^{z^1+d_{r^1}*}(b)$, $\tilde{V}^{z^2}_{r^2}(b) = V^{z^2}_{r^2}(b) + \chi^{z^2*}(b)$ for each $r^1 \in R^1$, $r^2 \in R^2$, $z^1, z^2 = 1,\dots,T$ and $b \in B$. Then, for each $i = 1,2$, (u^{i*}, λ^{i*}) is an optimal solution of the following linear program:

$$\min_{u^i,\lambda^i} \quad U^i = \sum_{m\in M} u^i_m + \sum_{z^i=1}^{T} \sum_{r^i\in R^i} k^{z^i*}_{r^i} \lambda^{z^i}_{r^i},$$

$$s.t. \quad \sum_{m\in b} u^i_m + \lambda^{z^i}_{r^i} \geq \tilde{V}^{z^i}_{r^i}(b), \quad \forall b \in B, \quad \forall r^i \in R^i, \quad \forall z^i = 1,\dots,T, \tag{40a}$$

$$u^i_m,\ \lambda^{z^i}_{r^i} \geq 0, \quad \forall m \in M, \quad \forall r^i \in R^i, \quad \forall z^i = 1,\dots,T. \tag{40b}$$

From the assumption of the mathematical induction, there exists edge price vector τ^{i*} such that (u^{i*}, τ^{i*}) is an optimal dual solution of the trip organization problem on the sub-network given $\tilde{V}$ value function for each $i = 1,2$:

$$\begin{aligned} \min_{u^i,\lambda^i} \quad & U = \sum_{m\in M} u^i_m + \sum_{t=1}^{T} \sum_{e\in E^i} q_e \tau^t_e, \\ s.t. \quad & \sum_{m\in b} u^i_m + \sum_{e\in r^i} \tau_e^{z^i+d_{r^i,e}} \geq \tilde{V}^{z^i}_{r^i}(b), \quad \forall b\in B, \quad \forall r^i \in R^i, \quad \forall z^i = 1,\dots,T, \\ & u^i_m,\ \tau^{t,i}_e \geq 0, \quad \forall m \in M, \quad \forall e \in E^i. \end{aligned} \tag{41}$$

Since the objective function (38) is the sum of the objective functions in (41) for $i = 1,2$, and the constraints are the combination of the constraints in the two linear programs, we know that $(u^{1*}, u^{2*}, \tau^{1*}, \tau^{2*}, \chi^*)$ must be an optimal solution of (38). We consider the edge price vector $\tau^* = (\tau^{1*}, \tau^{2*})$. Since $(u^{1*}, u^{2*}, \tau^{1*}, \tau^{2*}, \chi^*)$ satisfies constraints (38a) and (38b) and $u^*_m = u^{1*}_m + u^{2*}_m$ for all $m \in M$, (u^*, τ^*) is a feasible solution of (D) on the original network G. Furthermore, since (u^*, τ^*) achieves the same objective value as the optimal solution $(u^{1*}, u^{2*}, \tau^{1*}, \tau^{2*}, \chi^*)$ in (39), (u^*, τ^*) must be an optimal solution of (D) on the network G.

Finally, we conclude from cases 1 and 2 that in any series-parallel network, for any optimal solution (u^*, λ^*) of (Dk^*), there must exist a edge price vector τ^* such that (u^*, τ^*) is an optimal solution of (D). □

Proof of Lemma 10. We first proof that a utility vector $u^* \in U^*$ if and only if u^* is an optimal utility vector of the following linear program:

$$
\begin{aligned}
\min_{u,\lambda} \quad & \sum_{m\in M} u_m + \sum_{r\in R}\sum_{z=1}^{T-d_r} k_r^{z*}\lambda_r^z, \\
s.t. \quad & \sum_{m\in\bar{b}} u_m + \lambda_r^z \geq \overline{V}_r^z(\bar{b}), \quad \forall (\bar{b}, r, z) \in \overline{Trip}, && (\bar{\mathrm{D}}k^*.\mathrm{a}) \\
& u_m \geq 0, \ \lambda_r^z \geq 0, \quad \forall m \in M, \quad \forall z = 1, \dots T - d_r, \quad \forall r \in R. && (\bar{\mathrm{D}}k^*.\mathrm{b})
\end{aligned}
$$

We note that (u^*, λ^*) is a feasible solution of ($\bar{\mathrm{D}}k^*$) since for any $(\bar{b}, r, z) \in \overline{Trip}$,

$$
\sum_{m\in\bar{b}} u_m + \lambda_r^z \geq \sum_{m\in h_r^z(\bar{b})} u_m + \lambda_r^z \geq V_r^z(h_r^z(\bar{b})) = \overline{V}_r^z(\bar{b}),
$$

where $h_r^z(\bar{b})$ is a representative rider group of $\bar{b}$ given r and z.

Note that any (u^*, λ^*) is an optimal solution of (Dk^*) if and only if there exists any optimal solution x^* of (LPk^*), (x^*, u^*, λ^*), that satisfies the primal feasibility, dual feasibility, and complementary slackness conditions corresponding to (LPk^*) and (Dk^*). Given such x^*, we can construct $\bar{x}^*$ such that for any $r \in R$ and any $z = 1, \dots, T - d_r$, $\bar{x}_r^{z*}(\bar{b}) = x_r^{z*}(\bar{b})$ for all $\bar{b} \in B$, and $\bar{x}_r^{z*}(\bar{b}) = 0$ for all $\bar{b} \in \bar{B} \setminus B$. Such $\bar{x}^*$ is an optimal solution of ($\overline{\mathrm{LP}}k^*$) since it achieves the same total social value as x^*. It remains to show that $(\bar{x}^*, u^*, \lambda^*)$ satisfies the complementary slackness conditions associated with ($\overline{\mathrm{LP}}k^*$.a), ($\overline{\mathrm{LP}}k^*$.b), and ($\bar{\mathrm{D}}k^*$.a):

(1) We note that $\sum_{(\bar{b},z,r)\in\{\overline{Trip}|\bar{b}\ni m\}} \bar{x}_r^{z*}(\bar{b}) = \sum_{(b,z,r)\in\{Trip|b\ni m\}} x_r^{z*}(b)$ for any $m \in M$. From the complementary slackness condition associated with ($\overline{\mathrm{LP}}k^*$.a), we know that

$$
u_m^* \cdot \left(1 - \sum_{(\bar{b},z,r)\in\{\overline{Trip}|\bar{b}\ni m\}} \bar{x}_r^{z*}(\bar{b})\right) = u_m^* \cdot \left(1 - \sum_{(b,z,r)\in\{Trip|b\ni m\}} x_r^{z*}(b)\right) = 0, \quad \forall m \in M.
$$

Thus, the complementary slackness condition associated with (LPk^*.a) is satisfied.

(2) We note that $\sum_{\bar{b}\in\bar{B}} \bar{x}_r^{z*}(\bar{b}) = \sum_{b\in B} x_r^{z*}(\bar{b})$. From the complementary slackness condition associated with ($\overline{\text{LP}}k^*$.b), $\lambda_r^* \cdot \left(k_r^{z*} - \sum_{\bar{b}\in\bar{B}} \bar{x}_r^{z*}(\bar{b})\right) = \lambda_r^* \cdot \left(k_r^{z*} - \sum_{b\in B} x_r^{z*}(\bar{b})\right) = 0$. Therefore, the complementary slackness condition associated with (LPk^*.b) is satisfied.

(3) Since $\bar{x}^*$ is 0 for any $\bar{b} \in \bar{B} \setminus B$, we only need to check that (x^*, u^*, λ^*) satisfies the complementary slackness conditions associated with ($\bar{\text{D}}k^*$.a) for $(\bar{b}, r, z) \in Trip$. Since $\bar{x}_r^{z*}(\bar{b}) = x_r^{z*}(\bar{b})$ for all $\bar{b} \in B$, such complementary slackness conditions directly follow from that with respect to (x^*, u^*, λ^*).

We next show that any (u^*, λ^*) that is an optimal solution of ($\bar{\text{D}}k^*$) is also an optimal solution of (Dk^*). We note that $\bar{x}^*$ constructed from the optimal solution x^* in (LP) is also an optimal solution of ($\overline{\text{LP}}k^*$), and thus $(\bar{x}^*, u^*, \lambda^*)$ satisfies the complementary slackness conditions with respect to ($\overline{\text{LP}}k^*$.a), ($\overline{\text{LP}}k^*$.b), and ($\bar{\text{D}}k^*$.a). From the complementary slackness condition associated with ($\overline{\text{LP}}k^*$.a), we know that for any rider that is not assigned to a trip in $\bar{x}^*$, the utility is zero. Since the rider group assigned in x^* is the same as those in $\bar{x}^*$, we know that any rider that is not assigned to trips in x^* also has zero utility. We have $\sum_{m\in\bar{b}} u_m^* = \sum_{m\in h_r^z(\bar{b})} u_m^*$ for all $\bar{b} \in \bar{B}$, where $h_r^z(\bar{b})$ is the representative rider group that is organized given $\bar{b}$. Thus, (u^*, λ^*) is a feasible solution of (Dk^*).

Following the analogous argument as in (1) – (3), we can prove that (x^*, u^*, λ^*) also satisfies the complementary slackness conditions associated with ($\overline{\text{LP}}k^*$.a), ($\overline{\text{LP}}k^*$.b), and ($\bar{\text{D}}k^*$.a), where x^* is the optimal solution in ($\overline{\text{LP}}k^*$) that is used to construct $\bar{x}^*$.

Finally, following the proof of Lemma 3, we know that U^* is the set of equilibrium prices of goods in the equivalent economy. From Lemma 5, we know that the set of Walrasian equilibrium price is a lattice, and the maximum element is $u^\dagger$. Consequently, we can conclude that the set U^* is a lattice, and $u^\dagger$ is the maximum element. Since the optimal value of the objective function for all (u^*, τ^*) equals to $S(x^*)$, we can conclude that the total edge price given by $\tau^\dagger$ is no higher than that of any other equilibrium. $\square$

Proof of Proposition 2. Following the proofs of Theorem 1, we know that the socially optimal trip vector is computed in two steps: we first compute the temporally repeated flow following Algorithm 1, and then we compute the Walrasian equilibrium allocation in an equivalent economy. From Lemma 7, the Walrasian equilibrium allocation can be computed by the Kelso-Crawford algorithm. In Algorithm 3, we tackle the problem that the augmented trip value function $\bar{V}_r^z(\bar{b})$ as well as the representative rider group $h_r^z(\bar{b})$ are not readily known, and need to be iteratively computed.

In each iteration of Algorithm 1, the shortest route of the network is computed by Dijkstra algorithm in time $O(|N|^2)$, where $|N|$ is the number of nodes in the network. Since the capacity of at least one edge is completely allocated to the shortest route of every iteration, the number of iterations in Algorithm 1 is less than or equal to E. Therefore, the time complexity of step 1 is $O(|E||N|^2)$.

Additionally, in Algorithm 3, the time complexity of each iteration is $O(|M||L|)$, where $|L| = \sum_{r\in R^*} k_r^*(T - d_r)$. In each iteration, riders' utilities are non-decreasing and at least one rider increases their utility by ϵ. Since riders' utilities as in (2) can not exceed $V_{max} = \max_{m\in M} \alpha_m$, Algorithm 3 must terminate in $|M|V_{max}/\epsilon$ iterations. The time complexity of Algorithm 3 is $O\left(\frac{V_{max}}{\epsilon}|M|^2|L|\right)$. □

F Supplementary material for Section 5

Proof. Proof of Proposition 3 In the *edge-disjoint path* problem, one is given a network G, and k origin-destination pairs $(o_1, d_1), \dots, (o_k, d_k)$; the objective is to determine whether there exists a collection of k edge-disjoint paths $r_1, \dots, r_k$ where r_i is an o_i-d_i path in G. Vygen [1995] proved that the edge-disjoint path problem on directed graphs is NP complete. We show that solving (IP_{mult}) is NP-hard, via a reduction from the edge-disjoint path problem. Given an edge-disjoint path instance $(G, \{(o_i, d_i)\}_{i=1}^k)$, define k populations $M_1, \dots, M_k$; each population M_i has (o_i, d_i) as its origin-destination pair, and consists of a single agent who has $\alpha_i = 1$ and all other parameters equal to 0. If $T = 1$ and all edges have unit-capacity and 0 travel time cost, the optimal solution has a social welfare of k if and only if there exists k edge-disjoint paths $r_1, \dots, r_k$, where r_i is an o_i-d_i path. Thus, solving the edge-disjoint path problem becomes a special case of computing an optimal solution to (IP_{mult}), and hence the latter problem is NP-hard.

Proof. Proof of Proposition 4 Garg et al. [1993] proved that the integrality gap of the multi-commodity flow IP is $\Omega(\max\{k, \sqrt{|E|}\})$; we adapt their proof to show that (IP_{mult}) also has an integrality gap of $\Omega(\max\{k, \sqrt{|E|}\})$. Consider the following grid network, with origin-destination pairs $\{(s_1, t_1), \dots, (s_k, t_k)\}$; each point of intersection (marked as the red dot) is connected by an edge with capacity 1. All black edges in the following figure have a transit time of ε (which is a small positive number). While there are multiple routes for each origin-destination pair, let R_i be the route consisting of the line segment from s_i to $v_{i,i}$, the connecting edge at $v_{i,i}$, and the line segment from $v_{i,i}$ to t_i.

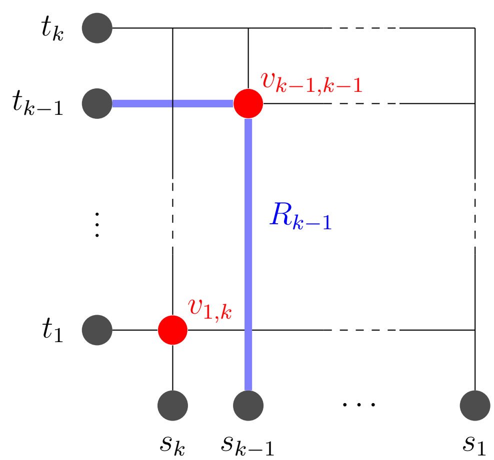


Figure 4: Example of a route R_{k-1} in the Grid network

We consider that $A = 1$ so that no trip sharing is allowed. Agents in the same population have the same origin and destination pair. For all agent m, $\theta_m = T$, $\alpha_m = 2$, and all other (unspecified) parameters are set to 0. The number of agents in each population i is larger or equal to T.

We next show that this problem instance has an integrality gap of $\Omega(k) = \Omega(\sqrt{|E|})$. First, we construct a feasible solution of the linear relaxation: For each population i, we send 1/2 agent to take route R_i at each time step $t = 1, \ldots, \lfloor T - k\varepsilon \rfloor$. This is a feasible solution since the flow on each edge of the grid is less than or equal to 1/2, and the flow on the edge of the intersection is less than or equal to 1. This solution has a total value of all trips $= k \cdot \lfloor T - k\varepsilon \rfloor$. So, the optimal value of the linear relaxation is at least $k \cdot \lfloor T - k\varepsilon \rfloor$.

Then, we construct an upper bound on the integer optimal value of trip organization. To begin with, we introduce a common origin s', and a common destination t' such that each s_i is connected to s' and each t_i is connected to t'. The transit time and capacity of each (s', s_i) and (t_i, t') edge is ε and 1, respectively. Consider the modified instance where all agents have the same origin s' and destination t', and the time horizon is increased to be $T + 2\varepsilon$, and the latest preferred arrival time $\theta_m = T + 2\varepsilon$ for all agents m. We argue that the value of an optimal integer solution of this modified problem instance is an upper bound on the optimal value of the original problem. Since all trips (when organized) have the same value of 2, the maximum trip value is achieved when the total number of trips that arrive before $T + 2\varepsilon$ is maximized. Therefore, the problem of computing the optimal trip organization reduces to the problem of computing the maximum integral flow over time that arrive before $T + 2\varepsilon$.

Since the revised problem instance has a single o-d pair, we know that there exists an optimal flow over time that is a temporally repeated flow (Ford Jr and Fulkerson [1958]). In our setting, this optimal temporally repeated flow is simply sending the maximum integral flow of the static problem for each time step $t = 1, \dots, \lfloor T+2\epsilon \rfloor$, i.e. sending one traveler in the population i through route R_i at each time step $t = 1, \dots, \lfloor T - k\varepsilon \rfloor$. Therefore, the optimal value of the integer solution in the modified problem instance is $2(T - k\varepsilon)$. Furthermore, we note that this optimal value is an upper bound of the optimal value of integral solution in the original problem since any integral flow that is feasible in the modified instance must also be feasible in the original problem instance, and adding the time cost ε to the edges $(s', s_i), (t_i, t')$ is canceled with augmenting the time horizon by 2ε. Therefore, we can conclude that the integrality gap of the original problem instance is at least $\frac{k \cdot \lfloor T - k\varepsilon \rfloor}{2\lfloor T - k\varepsilon \rfloor} = \Omega(k) = \Omega(\sqrt{|E|})$.

Furthermore, consider the following series-parallel network, with origin-destination pairs $\{(s_1, t_1), (s_2, t_2), (s_3, t_3), (s_4, t_4)\}$. All black edges have a travel time of ε, and a capacity of 1. The red edges, (t_1, t_2) and (s_1, s_2) have a travel time of T and a capacity of 1. Again, we assume that $A = 1$. Agents in the same population have the same origin and destination pair. Furthermore, $\theta_m = T$ for all agents m, $\alpha_m = 2$ for all $m \in M_1 \cup M_2$, $\alpha_m = 1$ for all $m \in M_3 \cup M_4$, where M_i is the set of agents associated with (s_i, t_i) for $i = 1, \dots, 4$. All other (unspecified) parameters are set to 0. Finally, we assume that the number of agents in each population M_i is at least T.

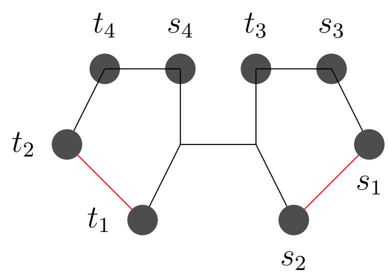


Figure 5: A Series-Parallel network with an integrality gap of $\frac{3(T-3\varepsilon)}{2T}$

Observe that for each population i, there is a unique route R_i that with travel time less than or equal to T. We note that the following fractional solution is feasible: For each population M_i, we send 1/2 of traveler to take route R_i at each time $t = 1, \dots, \lfloor T - 5\varepsilon \rfloor$ for $i = 1, 2$ and $t = 1, \dots, \lfloor T - \varepsilon \rfloor$ for $i = 3, 4$. This solution has a value of $1 \cdot \lfloor T - \varepsilon \rfloor + 2 \cdot \lfloor T - 5\varepsilon \rfloor$.

Following analogous procedure as that for the above grid network, we can show that the optimal integral solution is to organize one trip at each time step $t = 1, \dots, \lfloor T - \varepsilon \rfloor$ for population 3 and 4. No other trip can be further organized on this network. This optimal integral solution has a total trip value of $2\lfloor T - \varepsilon \rfloor$. From this problem instance, we can conclude that the integrality gap is at least $(\lfloor T - \varepsilon \rfloor + 2 \cdot \lfloor T - 5\varepsilon \rfloor)/2\lfloor T - \varepsilon \rfloor$, which approaches $\frac{3}{2}$ as $T \to \infty$. $\square$

We now provide a formal description of the Branch-and-Price algorithm described in

Section 5.

ALGORITHM 4: Branch and price algorithm for solving (IP_{mult})

Compute x^*, q^* as an optimal solution to the LP-relaxation of (IP_{mult}).
if q^* *is integral* **then**
  for $i \in I$ **do**
    Compute the optimal trip organization of submarket i with respect to capacity q^* using Algorithm 2.
  return *the optimal trip organization*
else
  Choose an arbitrary $i \in I, r \in R_i, t \in \{1, \ldots, T\}$ such that $q_r^{*i,t}$ is fractional
  Recursively call Algorithm 4 to compute $x^{(1)}$ as the optimal integer solution when constraint $q_r^{i,z} \leq \lfloor q_r^{*i,z} \rfloor$ is added, and $x^{(2)}$ as the optimal integer solution when constraint $q_r^{i,z} \geq \lceil q_r^{*i,z} \rceil$ is added.
  return $\arg\max\{S(x^{(1)}), S(x^{(2)})\}$

The branch and price algorithm start by computing an optimal solution of the linear relaxation of (IP_{mult}) (Line 1). If the optimal solution has an integral capacity allocation vector q^*, then we know that by using algorithm 3, we can compute the integral equilibrium trip organization vector and the associated edge prices and payments for each submarket (Line 3-6). If there exists at least one (i, r, z) such that $q_r^{*i,z}$ (the capacity allocated to population i on route r at time z) is fractional, we branch on the variable $q_r^{i,z}$ to create two sub-problems, where either $q_r^{i,z} \leq \lfloor q_r^{*i,z} \rfloor$ or $q_r^{i,z} \geq \lceil q_r^{*i,z} \rceil$ is added as an additional constraint. We resolve the linear relaxation associated with each subproblem, and continue to add additional constraints until we obtain an integer solution (Line 8-10). In our implementation, we also incorporate a pruning step – if the optimal value of the linear relaxation of a subproblem is smaller than the best integer solution that has been found, then we stop branching on that subproblem.

The key step of Algorithm (4) is to repeatedly compute the linear relaxation of the integer trip organization problem with additional constraints on q that has been added in the branching process. Any such linear program has exponential number of variables – the trip vector x. We compute the optimal (fractional) solution using column generation method. That is, we start by solving a restricted linear program that only includes a (small) subset of trips. Whether or not such solution is an optimal solution of the original LP can be verified by the dual feasibility. Recall that a violated dual constraint can be found in polynomial time using the greedy algorithm as in (Line 5-14 in Algorithm 2) due to the property that the augmented trip valuation function in each sub-market satisfies the gross substitutes condition. If we detect a violated constraint, we add the corresponding trip variable and resolve the primal problem; otherwise, we terminate with an optimal solution of the original LP.

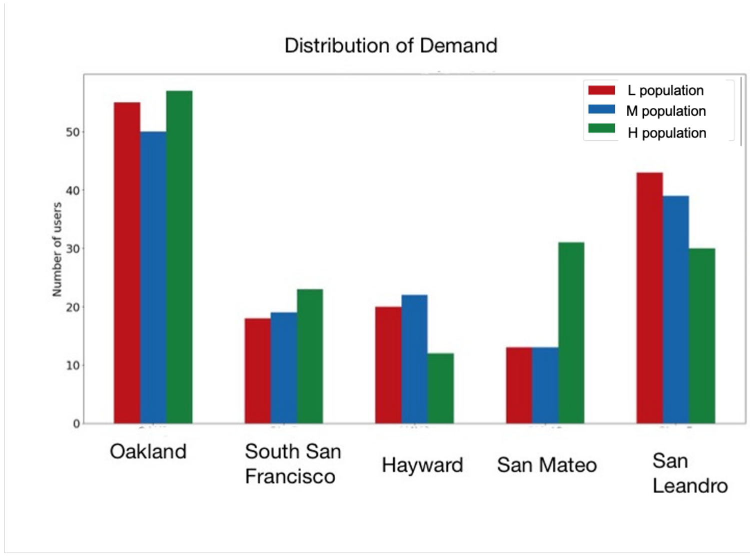
F.1 Carpooling and toll pricing in San Francisco Bay Area

The parameters of populations L, M, H are presented in the table below.

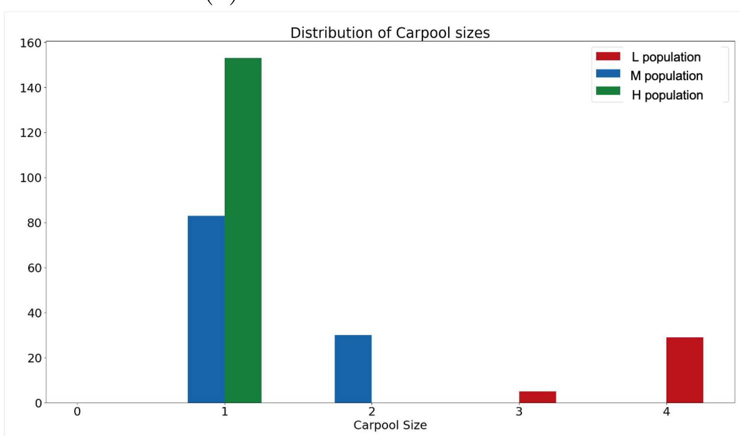
Parameter	Low	Medium
α_m	Uniformly distributed on $[30, 70]$	Uniformly distributed on $[80, 120]$
β_m	$\frac{10}{60}$	$\frac{30}{60}$
θ_m	Uniformly distributed on $[40, 60]$	Uniformly distributed on $[40, 60]$
ℓ_m	$\ell_m((z + d_r - \theta_m)_+) = (z + d_r - \theta_m)_+$	$\ell_m((z + d_r - \theta_m)_+) = (z + d_r - \theta_m)_+$
$\pi_m(\lvert b\rvert) + \gamma_m(\lvert b\rvert)d_r$	$\pi_L(\lvert b\rvert) = \begin{cases} 0.25(\lvert b\rvert - 1), & \lvert b\rvert \leq 5 \\ 0.5(\lvert b\rvert - 1), & 5 < \lvert b\rvert \leq 10 \\ \infty, & \lvert b\rvert > 10 \end{cases}$ $\gamma_m(\lvert b\rvert) = 0$	$\pi_M(\lvert b\rvert) = \begin{cases} 2(\lvert b\rvert - 1), & \lvert b\rvert \leq 3 \\ 4(\lvert b\rvert - 1), & \lvert b\rvert = 4 \\ \infty, & \lvert b\rvert > 4 \end{cases}$ $\gamma_m(\lvert b\rvert) = 0$

Parameter	High
α_m	Uniformly distributed on $[180, 220]$
β_m	$\frac{90}{60}$
θ_m	Uniformly distributed on $[40, 60]$
ℓ_m	$\ell_m((z + d_r - \theta_m)_+) = (z + d_r - \theta_m)_+$
$\pi_m(\lvert b\rvert) + \gamma_m(\lvert b\rvert)d_r$	$\pi_H(\lvert b\rvert) = \begin{cases} 4(\lvert b\rvert - 1), & \lvert b\rvert \leq 2 \\ 8(\lvert b\rvert - 1), & \lvert b\rvert = 3 \\ \infty, & \lvert b\rvert > 3 \end{cases}$ $\gamma_m(\lvert b\rvert) = 0$

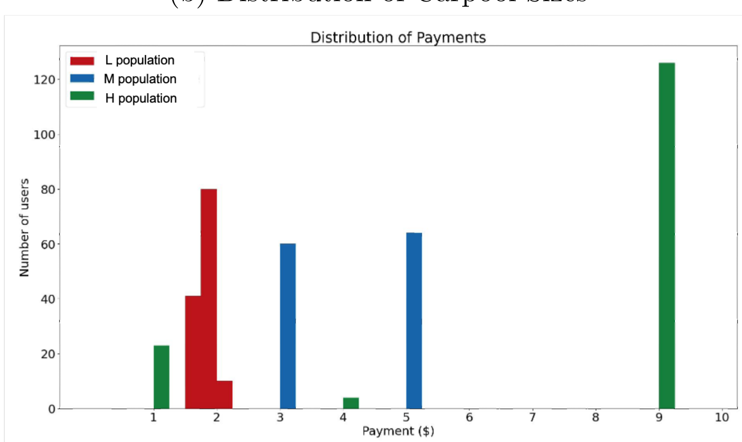
Five origin-destination pairs are considered in this instance, namely (Oakland, San Francisco), (South San Francisco, San Francisco), (Hayward, San Francisco), (San Mateo, San Francisco), and (San Leandro, San Francisco). We summarize the demand distribution, carpool sizes, and payments in the following figures:



(a) Distribution of Demand



(b) Distribution of Carpool Sizes



(c) Distribution of Payments

Figure 6