6.S979: Problem Set 2

Due: November 6, 2020

1. **SDP duality, Tsirelson's characterization, and NPA** In this problem we will explore the connection between the "primal" view of the NPA hierarchy in terms of extended correlation matrices, and the "dual" view in terms of sums of squares. We'll do this for the special case of the Tsirelson characterization for XOR games.

   Let $\mathcal{X}$ and $\mathcal{Y}$ be Alice and Bob's respective input sets. Recall that an extended correlation matrix C with rows and columns indexed by $\mathcal{X} \cup \mathcal{Y}$ is allowed by Tsirelson's characterization if

   - $C \succeq 0$ (i.e. C is a Hermitian, positive semidefinite matrix).
   - $C_{ii} = 1$ for all $i \in \mathcal{X} \cup \mathcal{Y}$.
   - $C_{xy} = C_{yx}$ for all x, y.

   (a) Suppose we are given an extended correlation matrix C. For any product PQ where $P, Q \in \{A_x\}_{x \in \mathcal{X}} \cup \{B_y\}_{y \in \mathcal{Y}}$, define the *pseudo-expectation* $\tilde{\mathbf{E}}[PQ]$ to be the value of the corresponding entry of C:

   $$\tilde{\mathbf{E}}[A_x A_{x'}] = C_{xx'}, \quad \tilde{\mathbf{E}}[A_x B_y] = \tilde{\mathbf{E}}[B_y A_x] = C_{xy}, \quad \tilde{\mathbf{E}}[B_y B_{y'}] = C_{yy'}.$$

   We can extend this by linearity to define the pseudo-expectation of any linear combination of such products. Given a polynomial of the form

   $$p = (\alpha A_x + \beta B_y)^\dagger (\alpha A_x + \beta B_y),$$

   show that there exists a vector v such that

   $$\tilde{\mathbf{E}}[p] = v^\dagger C v.$$

   (b) Suppose we have an SoS certificate that the bias of some XOR game is at most ν of the form

   $$\nu \cdot I - \sum_{x,y} s_{xy} A_x B_y = \sum_{i=1}^{k} r_i^\dagger r_i + \sum_x \alpha_x (I - A_x^2) + \sum_y \beta_y (I - B_y^2) + \sum_{x,y} \gamma_{x,y}([A_x, B_y]),$$

   where $s_{xy} \in \{\pm 1\}$, each r_i is a linear combination of A_x and B_y operators, and $\alpha_x, \beta_y, \gamma_{xy}$ are complex numbers. Show that this implies that for every correlation matrix C satisfying Tsirelson's criteria,

   $$\tilde{\mathbf{E}}[\sum_{x,y} s_{xy} A_x B_y] \leq \nu.$$

This is known as "weak duality": it says that any SoS certificate of this form also upper-bounds the value attained by a Tsirelson correlation. In fact, it is true (but you need not prove) that "strong duality" holds: the optimal game value attained for a Tsirelson correlation is equal to the optimal upper-bound that can be proven by an SoS certificate of this form.

2. **Embezzlement and Schmidt coefficients** In this problem we will see a fun example of the utility of Schmidt coefficients. We consider the task of *embezzlement of entanglement*, introduced by van Dam and Hayden. In this setting, we imagine that Alice and Bob go to the entanglement bank to get a state $|\psi\rangle_{AB}$. They each perform a local operation on the state and their local registers, and then send the state back to the bank. Alice and Bob's goal is to extract one EPR pair of entanglement while modifying the bank's state as little as possible, i.e. to carry out the transformation

$$|\psi\rangle_{AB} \otimes |0\rangle_{A'} \otimes |0\rangle_{B'} \xrightarrow{V^A \otimes V^B} |\psi\rangle_{AB} \otimes \frac{1}{\sqrt{2}}(|0\rangle_{A'} \otimes |0\rangle_{B'} + |1\rangle_{A'} \otimes |1\rangle_{B'}),$$

where V^A is an isometry acting only on AA' and V^B is an isometry acting only on BB'.

   (a) Suppose the Schmidt coefficients of $|\psi\rangle$ are $\sigma_1, \ldots, \sigma_k$ for some $k < \infty$. Write the Schmidt coefficients of the joint states on $AA'BB'$ before and after the embezzlement transformation.

   (b) Is embezzlement possible for finite k? Why or why not?

3. **Testing commutation:** In this problem we're going to analyze a game to test that two measurements approximately commute. This is very useful for analyzing MIP* proofs. In the basic commutation test, Alice is sent the question 0 asked for two answers a_0, a_1. Bob is sent a bit $y \in \{0,1\}$, and responds with an answer b. The players win the test if $a_y = b$. We denote the players' shared stated by $|\psi\rangle$, Alice's measurement elements by $\{A_{a_0,a_1}\}$ and Bob's by $\{B^y_b\}$.

   (a) Write an expression for the success probability of the players in the test. (Hint: it should look like a sum of terms of the form $\langle\psi| A \otimes B |\psi\rangle$ for some operators A, B.)

   (b) Suppose Alice and Bob win the game with certainty. Prove that for any y,

$$\sum_{a_0,a_1} A_{a_0,a_1} \otimes B^y_{a_y} |\psi\rangle = |\psi\rangle .$$

   (c) Deduce that

$$A_{a_0,a_1} \otimes I |\psi\rangle = A_{a_0,a_1} \otimes B^0_{a_0} |\psi\rangle = \sum_{a_0'} A_{a_0',a_1} \otimes B^0_{a_0} |\psi\rangle$$
$$A_{a_0,a_1} \otimes I |\psi\rangle = A_{a_0,a_1} \otimes B^1_{a_1} |\psi\rangle = \sum_{a_1'} A_{a_0,a_1'} \otimes B^1_{a_1} |\psi\rangle$$

   Hint: use the previous part together with orthogonality between elements corresponding to different outcomes (e.g. the fact that $A_{a_0,a_1} A_{a_0',a_1} = 0$ for $a_0 \neq a_0'$).

(d) Using the previous two parts, prove that

$$A_{a_0,a_1} \otimes I \left|\psi\right\rangle = I \otimes B^0_{a_0} B^1_{a_1} \left|\psi\right\rangle = I \otimes B^1_{a_1} B^0_{a_0} \left|\psi\right\rangle .$$

(e) Optional bonus: what about the case where Alice and Bob win with probability $1 - \epsilon$? Can you make a quantitative version of the preceding arguments work?

4. **Non-signalling correlations:** The NPA hierarchy gives us a collection of outer approximations to the set $\mathcal{C}_{qc}$ of quantum commuting correlations. A much cruder outer approximation is the set of *non-signalling correlations* $\mathcal{C}_{ns}$. A correlation $p(a, b|x, y)$ is non-signalling if for every a, x, y, y'

$$p(a|x, y) = p(a|x, y'),$$

and likewise for every b, x, x', y

$$p(b|x, y) = p(b|x', y).$$

That is, Alice's outcome probabilities for a given question should be the same regardless of Bob's question, and vice versa.

(a) Prove that $\mathcal{C}_{qc} \subseteq \mathcal{C}_{ns}$. Recall that a correlation is in $\mathcal{C}_{qc}$ if it can be written as

$$p(a, b|x, y) = \left\langle\psi\right| A^x_a B^y_b \left|\psi\right\rangle ,$$

for some state $\left|\psi\right\rangle$ and Hermitian operators A^x_a, B^y_b satisfying the conditions

$$\begin{aligned}
&\forall x, (A^x_a)^2 = A^x_a \quad \text{and} \quad \sum_a A^x_a = I \\
&\forall y, (B^y_b)^2 = B^y_b \quad \text{and} \quad \sum_b B^y_b = I \\
&\forall x, y, a, b,\ [A^x_a, B^y_b] = 0.
\end{aligned}$$

(b) Show that there is a non-signalling correlation that wins the CHSH game with certainty.